



CFL3D has several turbulence model capabilities. Appendix H provides the derivation for the advanced turbulence model equations. Be aware that while some variables in this appendix are consistent with the rest of the manual (and are listed in the Nomenclature section), many are introduced, defined, and used only in these sections and may or may not appear in the Nomenclature listing. Also, for simplicity's sake in this Appendix, $Re = Re_{\tilde{L}_R}$.

H.1 Equations of Motion

Following Wilcox⁴⁶, Favre¹⁹ averaging can be used with the Navier-Stokes equations to account for turbulent fluctuations. The resulting equations of motion can be written using the summation convention as follows. The full Navier-Stokes equations are shown here, but in CFL3D, they are solved as the thin-layer approximation in pre-selected coordinate direction(s). The \sim indicates a dimensional quantity.

$$\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}_j}(\tilde{\rho} \tilde{u}_j) = 0 \quad (\text{H-1})$$

$$\frac{\partial}{\partial \tilde{t}}(\tilde{\rho} \tilde{u}_i) + \frac{\partial}{\partial \tilde{x}_j}(\tilde{\rho} \tilde{u}_j \tilde{u}_i) = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{\partial \tilde{\tau}_{ji}}{\partial \tilde{x}_j} \quad (\text{H-2})$$

$$\frac{\partial}{\partial \tilde{t}}(\tilde{\rho} \tilde{E}) + \frac{\partial}{\partial \tilde{x}_j}(\tilde{\rho} \tilde{u}_j \tilde{H}) = \frac{\partial}{\partial \tilde{x}_j}[\tilde{u}_i \tilde{\tau}_{ij} - \tilde{q}_j + \tilde{\psi}_j] \quad (\text{H-3})$$

where

$$\tilde{p} = (\gamma - 1) \left[\tilde{\rho} \tilde{E} - \frac{1}{2} \tilde{\rho} (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2) \right] \quad (\text{H-4})$$

$$\tilde{E} = \tilde{e} + \frac{1}{2} (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2) \quad (\text{H-5})$$

$$\tilde{H} = \tilde{E} + \frac{\tilde{p}}{\tilde{\rho}} \quad (\text{H-6})$$

$$\tilde{q}_j = -\frac{1}{\gamma - 1} \left(\frac{\tilde{\mu}}{\text{Pr}} + \frac{\tilde{\mu}_T}{\text{Pr}_T} \right) \frac{\partial \tilde{a}^2}{\partial \tilde{x}_j} \quad (\text{H-7})$$

$$\tilde{a}^2 = \frac{\gamma \tilde{p}}{\tilde{\rho}} \quad (\text{H-8})$$

Note that the kinetic energy of the fluctuating turbulent field \tilde{k} is ignored in the definition of \tilde{E} in CFL3D (it is assumed that $\tilde{k} \ll \tilde{e} + \frac{1}{2}(\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2)$). Define

$$\begin{aligned} \tilde{S}_{ij} &= \frac{1}{2} \left(\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} + \frac{\partial \tilde{u}_j}{\partial \tilde{x}_i} \right) \\ \tilde{W}_{ij} &= \frac{1}{2} \left(\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} - \frac{\partial \tilde{u}_j}{\partial \tilde{x}_i} \right) \end{aligned} \quad (\text{H-9})$$

Also, the magnitude of vorticity is

$$\tilde{\Omega} = \sqrt{2 \tilde{W}_{ij} \tilde{W}_{ij}} \quad (\text{H-10})$$

The shear stress terms $\tilde{\tau}_{ij}$ is composed of a laminar and a turbulent component as

$$\tilde{\tau}_{ij} = \tilde{\tau}_{ij}^L + \tilde{\tau}_{ij}^T \quad (\text{H-11})$$

where

$$\tilde{\tau}_{ij}^L = 2\tilde{\mu} \left(\tilde{S}_{ij} - \frac{1}{3} \frac{\partial \tilde{u}_k}{\partial \tilde{x}_k} \delta_{ij} \right) \quad (\text{H-12})$$

For all *eddy-viscosity* models in CFL3D, the following approximations are made:

$$\tilde{\tau}_{ij}^T = 2\tilde{\mu}_T \left(\tilde{S}_{ij} - \frac{1}{3} \frac{\partial \tilde{u}_k}{\partial \tilde{x}_k} \delta_{ij} \right) \quad (\text{H-13})$$

$$\tilde{\psi}_j = 0 \quad (\text{H-14})$$

Note that $\frac{\tilde{\tau}_{ij}^T}{\tilde{\rho}} = -\tilde{\tau}_{ij}^T / \tilde{\rho}$, or $-2 \frac{\tilde{\mu}_T}{\tilde{\rho}} \left(\tilde{S}_{ij} - \frac{1}{3} \frac{\partial \tilde{u}_k}{\partial \tilde{x}_k} \delta_{ij} \right)$ in this case.

Currently, for the *nonlinear* models in CFL3D,

$$\tilde{\tau}_{ij}^T = 2\tilde{\mu}_T \left(\tilde{S}_{ij} - \frac{1}{3} \frac{\partial \tilde{u}_k}{\partial \tilde{x}_k} \delta_{ij} \right) - \frac{2}{3} \tilde{\rho} \tilde{k} \delta_{ij} + \quad (\text{H-15})$$

$$2\tilde{\mu}_T K_1 \frac{\tilde{k}}{\tilde{\epsilon}} (\tilde{S}_{ik} \tilde{W}_{kj} - \tilde{W}_{ik} \tilde{S}_{kj}) -$$

$$2\tilde{\mu}_T K_2 \frac{\tilde{k}}{\tilde{\epsilon}} \left(\tilde{S}_{ik} \tilde{S}_{kj} - \frac{1}{3} \tilde{S}_{kl} \tilde{S}_{lk} \delta_{ij} \right)$$

$$\tilde{\psi}_j = \left(\tilde{\mu} + \frac{\tilde{\mu}_T}{\sigma_k} \right) \frac{\partial \tilde{k}}{\partial \tilde{x}_j} \quad (\text{H-16})$$

where $\tilde{k}/\tilde{\epsilon}$ is replaced by $1/\tilde{\omega}$ when the $k - \omega$ equations rather than the $k - \epsilon$ equations are employed.

The Navier-Stokes equations are nondimensionalized and written in generalized coordinates, as described Appendix F. For *eddy-viscosity* models, the end result is that the turbulent Navier-Stokes equations are identical to the laminar equations with the exception that

$$\tilde{\mu} \text{ is replaced by } \tilde{\mu} + \tilde{\mu}_T$$

and $\frac{\tilde{\mu}}{Pr}$ is replaced by $\frac{\tilde{\mu}}{Pr} + \frac{\tilde{\mu}_T}{Pr_T}$

where $\tilde{\mu}_T$ is the eddy viscosity value obtained by whatever turbulence model is used, and $Pr = 0.72$, $Pr_T = 0.9$. For the *nonlinear* models, both the above substitutions must be made *and* the following additions as well. The term

$$-\frac{2}{3} \tilde{\rho} \tilde{k} \delta_{ij} + 2\tilde{\mu}_T K_1 \frac{\tilde{k}}{\tilde{\epsilon}} (\tilde{S}_{ik} \tilde{W}_{kj} - \tilde{W}_{ik} \tilde{S}_{kj}) - 2\tilde{\mu}_T K_2 \frac{\tilde{k}}{\tilde{\epsilon}} \left(\tilde{S}_{ik} \tilde{S}_{kj} - \frac{1}{3} \tilde{S}_{kl} \tilde{S}_{lk} \delta_{ij} \right)$$

is added to $\tilde{\tau}_{ij}$ in the momentum and energy equations and

$$\frac{\partial}{\partial \tilde{x}_j} \left[\left(\tilde{\mu} + \frac{\tilde{\mu}_T}{\sigma_k} \right) \frac{\partial \tilde{k}}{\partial \tilde{x}_j} \right]$$

is added to the energy equation. These additions are made in subroutines `gfluxv`, `hfluxv`, and `ffluxv`.

H.2 Nondimensionalizations

The turbulence equations are nondimensionalized by the same reference quantities as the Navier-Stokes equations: \tilde{a}_∞ , $\tilde{\rho}_\infty$, $\tilde{\mu}_\infty$, and $\tilde{L}_R = \tilde{L}/L_{ref}$. The nondimensionalized variables used with the turbulence models are:

$$\begin{aligned}
 k &= \frac{\tilde{k}}{\tilde{a}_\infty^2} & \omega &= \frac{\tilde{\mu}_\infty \tilde{\omega}}{\tilde{\rho}_\infty \tilde{a}_\infty^2} & P_k &= \frac{\tilde{P}_k \tilde{L}_R^2}{\tilde{\mu}_\infty \tilde{a}_\infty^2} \\
 \varepsilon &= \frac{\tilde{\mu}_\infty \tilde{\varepsilon}}{\tilde{\rho}_\infty \tilde{a}_\infty^4} & \rho &= \frac{\tilde{\rho}}{\tilde{\rho}_\infty} & P_\omega &= \frac{\tilde{P}_\omega \tilde{L}_R^2}{\tilde{\rho}_\infty \tilde{a}_\infty^2} \\
 u &= \frac{\tilde{u}}{\tilde{a}_\infty} & p &= \frac{\tilde{p}}{\tilde{\rho}_\infty \tilde{a}_\infty^2} & P_\varepsilon &= \frac{\tilde{P}_\varepsilon \tilde{L}_R^2}{\tilde{\rho}_\infty \tilde{a}_\infty^4} \\
 x &= \frac{\tilde{x}}{\tilde{L}_R} & \mu &= \frac{\tilde{\mu}}{\tilde{\mu}_\infty} & \Omega &= \frac{\tilde{\Omega} \tilde{L}_R}{\tilde{a}_\infty} \\
 \tau_{ij} &= \frac{\tilde{\tau}_{ij} \tilde{L}_R}{\tilde{\mu}_\infty \tilde{a}_\infty} & t &= \frac{\tilde{t} \tilde{a}_\infty}{\tilde{L}_R}
 \end{aligned} \tag{H-17}$$

H.3 Zero-equation Models

H.3.1 Baldwin-Lomax Model

(**ivisc** = 2)

The Baldwin-Lomax¹⁰ model is not a field-equation model; it is an algebraic model. Because it is the original model employed in CFL3D, its implementation differs in many respects from the more advanced models. First of all, it does not use the *minimum distance function* as its length scale, like the other models. Instead, it uses a *directed distance* from the $i = 1, j = 1, k = 1, i = \mathbf{idim}, j = \mathbf{jdim}$, or $k = \mathbf{kdim}$ point along a constant index line. For example, if the “body” is at $k = 1$, then the directed distance to a given point in the field is the directed (normal) distance from k_{given} to $k = 1$, keeping i and j fixed. See Figure H-1(a). The directed distance is

$$d = \vec{r}_k \cdot \vec{n} - \vec{r}_0 \cdot \vec{n} \tag{H-18}$$

where \vec{r}_k is the vector from the origin to k_{given} and \vec{r}_0 is the vector from the origin to the $k = 1$ point. Note that if grid lines curve significantly d can become negative as in Figure H-1(b). If the grid lines do this, CFL3D currently does not allow the distance to go

negative. Instead, it computes d out to its maximum, then sets all distances thereafter to that maximum value. However, if the grid behaves like the grid in Figure H-1(b), the Baldwin-Lomax model, which is inherently dependent on the grid structure, is probably not a good choice anyway. The use of any of the other field-equation models is recommended instead.

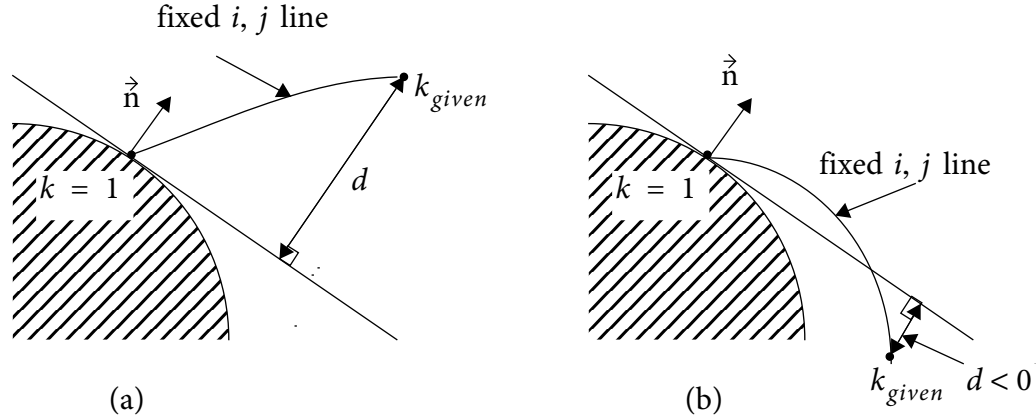


Figure H-1. Directed distance schematic.

In the current implementation in CFL3D, if the grid contains multiple blocks, Baldwin-Lomax should only be implemented on blocks that contain bodies. This restriction applies to the Baldwin-Lomax model only. It does *not* apply to the more general field-equation models.

For the Baldwin-Lomax model,

$$\begin{aligned} \mu_T &= \mu_{T, inner} & y \leq y_{crossover} \\ \mu_T &= \mu_{T, outer} & y > y_{crossover} \end{aligned} \quad (\text{H-19})$$

where $y_{crossover}$ is the location along the constant index line where $\mu_{T, inner}$ exceeds $\mu_{T, outer}$, marching away from the wall. The inner eddy viscosity is

$$\mu_{T, inner} = \rho l^2 \Omega \left(\frac{Re}{M_\infty} \right) \quad (\text{H-20})$$

where Ω is the magnitude of the vorticity and

$$l = y[1 - \exp(-y^+/26)] \quad (\text{H-21})$$

Since it now appears to be fairly widely accepted that the definition of y^+ for the damping term needs to be modified in Baldwin-Lomax to give better answers when there are large temperature gradients near the wall,

$$y^+ = \frac{\sqrt{\tilde{\rho}\tilde{\tau}_w}\tilde{y}}{\tilde{\mu}} \quad (\text{H-22})$$

is used rather than

$$y^+ = \frac{\sqrt{\tilde{\rho}_w\tilde{\tau}_w}\tilde{y}}{\tilde{\mu}_w} \quad (\text{H-23})$$

Nondimensionally, this becomes

$$y^+ = \frac{\sqrt{\rho\tau_w}}{\mu} y \left(\frac{Re}{M_\infty} \right)^{\frac{1}{2}} \quad (\text{H-24})$$

The outer eddy viscosity is

$$\mu_{T, outer} = 0.0168(1.6)\rho F_{wake} F_{kleb} \left(\frac{Re}{M} \right) \quad (\text{H-25})$$

where

$$F_{wake} = \min[y_{max} F_{max}, 1.0 y_{max} u_{dif}^2 / F_{max}] \quad (\text{H-26})$$

$$F(y) = y\Omega[1 - \exp(-y^+/26)]$$

In wakes, $\exp(-y^+/26)$ is set to zero. F_{max} is the maximum value of $F(y)$ that occurs in a profile and y_{max} is the value at which F_{max} occurs. Also,

$$F_{kleb} = \left[1 + 5.5 \left(\frac{0.3y}{y_{max}} \right)^6 \right]^{-1} \quad (\text{H-27})$$

$$u_{dif} = (\sqrt{u^2 + v^2 + w^2})_{max} - (\sqrt{u^2 + v^2 + w^2})_{min}$$

The second term in u_{dif} is taken to be zero, except in wakes. The region for the search for F_{max} is currently bound by

$$0.2\eta_{max} + 1 < \eta < 0.8\eta_{max} + 1 \quad (\text{H-28})$$

where η , in this case, is the index direction away from the body.

H.3.2 Baldwin-Lomax with Degani-Schiff Modification

(ivisc = 3)

The Degani-Schiff¹⁶ modification to the Baldwin-Lomax model is an algorithmic change which attempts to select the *first* occurrence of F_{max} in a search from the wall outwards. This can be important when there is a vortex somewhere above the body surface. If the code is not forced to select the F_{max} in the boundary layer, it may choose a length scale corresponding to the distance to the vortex, since often F can be larger in the vortex.

The test in CFL3D is very simple-minded and can often fail. However, it is quite difficult to find a test that works for all circumstances; so, for lack of anything better, the following is used. Marching outward away from the body, F_{max} is updated index by index. Then, if

$$F < 0.9F_{max}, \quad (\text{H-29})$$

the code stops searching.

H.4 One- and Two-equation Field-equation Models

All of the one- and two-equation model equations can be written in the general form

$$\frac{\partial}{\partial t}(X) + u_j \frac{\partial}{\partial x_j}(X) = S_p + S_D + D \quad (\text{H-30})$$

where S_p is a “production” source term(s), S_D is a “destruction” source term(s), and D represents diffusion terms of the form $\frac{\partial}{\partial x_j} \left[() \cdot \frac{\partial X}{\partial x_j} \right]$. Note that S_p and S_D have been grouped together rather loosely. In Spalart’s model in CFL3D, for example, part of Spalart’s production term is grouped in S_D for convenience. In Menter’s model in CFL3D, the cross-derivative term is grouped in S_D . In CFL3D, the S_p terms are treated explicitly while the S_D terms are linearized and treated implicitly. This is Spalart’s “third strategy.”³⁵

All of the field-equation models are solved *uncoupled* from the Navier-Stokes equations. All of the models are solved in essentially the same fashion. Details are given in “Solution Method” on page 299 in the form of an example.

Also, all of the one- and two-equation models are based on incompressible turbulence equations. No compressibility corrections have been added. Hence, for problems where the turbulent Mach number, $M_T = \sqrt{2\tilde{k}/\tilde{a}^2}$, is high, these turbulence models may not be applicable. Note that for most subsonic through low supersonic aerodynamic applications, the incompressible forms of the turbulence models are generally expected to be valid.

All of the field equation models except for Wilcox $k - \omega$ make use of the *minimum distance function*, or the distance to the nearest wall, s_{\min} . This distance differs from the *directed distance* used by the Baldwin-Lomax model in that it does *not* follow grid lines and can be computed across zone boundaries. Hence, it is more easily applicable to multiple zone applications. The minimum distance function represents the distance to the nearest viscous wall and takes into account grid skewness when computing the nearest wall-point location (in subroutine `findmin_new`). The exception is the Baldwin-Barth model which uses a minimum distance algorithm (subroutine `findmin`) that does not take grid skewness into account. (The reason for this exception is that the Baldwin-Barth model requires other variables not currently implemented in subroutine `findmin_new`.)

H.4.1 Baldwin-Barth Model

(`ivisc` = 4)

The Baldwin-Barth⁹ model solves a single field equation for a turbulent Reynolds number term R :

$$\frac{\partial R}{\partial t} + u_j \frac{\partial R}{\partial x_j} = (C_{\varepsilon_2} f_2 - C_{\varepsilon_1}) \sqrt{RP} + \frac{M_\infty}{Re} \left(v + \frac{v_T}{\sigma_\varepsilon} \right) \frac{\partial^2 R}{\partial x_j^2} - \frac{M_\infty}{Re} \frac{1}{\sigma_\varepsilon} \frac{\partial}{\partial x_j} \left(v_T \frac{\partial R}{\partial x_j} \right) \quad (\text{H-31})$$

where

$$\frac{1}{\sigma_\varepsilon} = (C_{\varepsilon_2} - C_{\varepsilon_1}) \frac{\sqrt{C_\mu}}{\kappa^2} \quad (\text{H-32})$$

$$v_T = C_\mu R D_1 D_2 \quad (\text{H-33})$$

$$D_1 = 1 - \exp\left(-\frac{y^+}{A^+}\right) \quad (\text{H-34})$$

$$D_2 = 1 - \exp\left(-\frac{y^+}{A_2^+}\right) \quad (\text{H-35})$$

$$f_2 = \frac{C_{\varepsilon_1}}{C_{\varepsilon_2}} + \left(1 - \frac{C_{\varepsilon_1}}{C_{\varepsilon_2}}\right) \left(\frac{1}{\kappa y^+} + D_1 D_2\right) \left\{ \sqrt{D_1 D_2} + \right. \quad (\text{H-36})$$

$$\left. \frac{y^+}{\sqrt{D_1 D_2}} \left[\frac{1}{A^+} \exp\left(-\frac{y^+}{A^+}\right) D_2 + \frac{1}{A_2^+} \exp\left(-\frac{y^+}{A_2^+}\right) D_1 \right] \right\}$$

$$\mu_T = \rho \nu_T \quad (\text{H-37})$$

$$\begin{array}{lll} \kappa = 0.41 & C_{\varepsilon_1} = 1.2 & C_{\varepsilon_2} = 2.0 \\ C_\mu = 0.09 & A^+ = 26 & A_2^+ = 10 \end{array} \quad (\text{H-38})$$

The production term P is given by

$$P = 2\nu_T S_{ij} S_{ij} - \frac{2}{3} \nu_T \left(\frac{\partial u_k}{\partial x_k} \right)^2 \quad (\text{H-39})$$

but is approximated as

$$P \cong \nu_T \Omega^2 \quad (\text{H-40})$$

Hence, using the general form in Equation (H-30) with $X = R$,

$$S_P \cong (C_{\varepsilon_2} f_2 - C_{\varepsilon_1}) \sqrt{C_\mu D_1 D_2} \Omega R \quad (\text{H-41})$$

$$D = \frac{M_\infty}{Re} \left(\nu + \frac{\nu_T}{\sigma_\varepsilon} \right) \frac{\partial^2 R}{\partial x_j^2} - \frac{M_\infty}{Re} \frac{1}{\sigma_\varepsilon} \frac{\partial}{\partial x_j} \left(\nu_T \frac{\partial R}{\partial x_j} \right) \quad (\text{H-42})$$

Since the Baldwin-Barth model requires y^+ (rather than just the minimum distance to the wall), additional information about the nearest wall point needs to be stored as well as the minimum distance function d . For simplicity, it is currently assumed that in the regions where y^+ has an effect, these additional wall values are in the same grid zone as the point in question.

H.4.2 Spalart-Allmaras Model

(ivisc = 5)

The Spalart-Allmaras³⁴ model solves a single field equation for a variable $\hat{\nu}$ related to the eddy viscosity through

$$\mu_T = \rho \hat{v} f_{v_1} \quad (\text{H-43})$$

where

$$f_{v_1} = \frac{\chi^3}{\chi^3 + C_{v_1}^3} \quad (\text{H-44})$$

$$\chi \equiv \frac{\hat{v}}{\nu} \quad (\text{H-45})$$

The equation is

$$\begin{aligned} \frac{\partial \hat{v}}{\partial t} + u_j \frac{\partial \hat{v}}{\partial x_j} = & C_{b_1} [1 - f_{t_2}] \Omega \hat{v} \\ & + \frac{M_\infty}{Re} \left\{ C_{b_1} [(1 - f_{t_2}) f_{v_2} + f_{t_2}] \frac{1}{\kappa^2} - C_{w_1} f_w \right\} \left(\frac{\hat{v}}{d} \right)^2 \\ & - \frac{M_\infty C_{b_2}}{Re \sigma} \hat{v} \frac{\partial^2 \hat{v}}{\partial x_j^2} \\ & + \frac{M_\infty}{Re \sigma} \frac{1}{\partial x_j} \left[(\nu + (1 + C_{b_2}) \hat{v}) \frac{\partial \hat{v}}{\partial x_j} \right] \end{aligned} \quad (\text{H-46})$$

(Note that Spalart's trip function is not used.) Also,

$$f_{t_2} = C_{t_3} \exp(-C_{t_4} \chi^2) \quad (\text{H-47})$$

$$d = \text{distance to the closest wall} = \text{minimum distance function} \quad (\text{H-48})$$

$$f_w = g \left[\frac{1 + C_{w_3}^6}{g^6 + C_{w_3}^6} \right]^{\frac{1}{6}} = \left[\frac{g^{-6} + C_{w_3}^{-6}}{1 + C_{w_3}^{-6}} \right]^{-\frac{1}{6}} \quad (\text{H-49})$$

$$g = r + C_{w_2} (r^6 - r) \quad (\text{H-50})$$

$$r = \frac{\hat{v}}{\hat{S} \left(\frac{Re}{M_\infty} \right) \kappa^2 d^2} \quad (\text{H-51})$$

where

$$\hat{S} = \Omega + \frac{\hat{v} f_{v_2}}{\left(\frac{Re}{M_\infty}\right) \kappa^2 d^2} \quad (\text{H-52})$$

$$f_{v_3} = \text{no longer used} \quad (\text{H-53})$$

$$f_{v_2} = 1 - \frac{\chi}{1 + \chi f_{v_1}} \quad (\text{H-54})$$

CFL3D currently uses Spalart's Version Ia³⁴. The fv3 term was employed as a smooth fix to prevent \hat{S} from going negative prior to 12/97, but was removed after an error was discovered (\hat{S} and fv2 were also different). The constants are

$$\begin{aligned} C_{b_1} &= 0.1355 & \sigma &= \frac{2}{3} & C_{b_2} &= 0.622 & \kappa &= 0.41 & C_{w_2} &= 0.3 \\ C_{w_3} &= 2.0 & C_{v_1} &= 7.1 & C_{t_3} &= 1.2 & C_{t_4} &= 0.5 & & \\ C_{w_1} &= \frac{C_{b_1}}{\kappa^2} + \frac{(1 + C_{b_2})}{\sigma} & & & & & & & & \end{aligned} \quad (\text{H-55})$$

(note typo in ref. 34)

For the general form in Equation (H-30):

$$X = \hat{v} \quad (\text{H-56})$$

$$S_p = C_{b_1} [1 - f_{t_2}] \Omega \hat{v} \quad (\text{H-57})$$

$$S_D = \frac{M_\infty}{Re} \left\{ C_{b_1} [(1 - f_{t_2}) f_{v_2} + f_{t_2}] \frac{1}{\kappa^2} - C_{w_1} f_w \right\} \left(\frac{\hat{v}}{d} \right)^2 \quad (\text{H-58})$$

$$D = - \frac{M_\infty}{Re} \frac{C_{b_2}}{\sigma} \hat{v} \frac{\partial^2 \hat{v}}{\partial x_j^2} + \frac{M_\infty}{Re} \frac{1}{\sigma} \frac{\partial}{\partial x_j} \left[(v + (1 + C_{b_2}) \hat{v}) \frac{\partial \hat{v}}{\partial x_j} \right] \quad (\text{H-59})$$

Note that in CFL3D, the S_p and S_D terms are grouped differently than Spalart's. Part of Spalart's production term is grouped into S_D because it has the common factor M_∞/Re , like the other destruction terms.

H.4.3 Wilcox k-Omega Model

(ivisc = 6)

For Wilcox's⁴⁶ model,

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{1}{\rho} P_k \left(\frac{M_\infty}{Re} \right) - \beta' k \omega \left(\frac{Re}{M_\infty} \right) + \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_T}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] \left(\frac{M_\infty}{Re} \right) \quad (\text{H-60})$$

$$\frac{\partial \omega}{\partial t} + u_j \frac{\partial \omega}{\partial x_j} = \frac{1}{\rho} P_\omega \left(\frac{M_\infty}{Re} \right) - \beta \omega^2 \left(\frac{Re}{M_\infty} \right) + \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_T}{\sigma_\omega} \right) \frac{\partial \omega}{\partial x_j} \right] \left(\frac{M_\infty}{Re} \right) \quad (\text{H-61})$$

In this model, C_μ is incorporated into the definition of ω , so

$$\mu_T = \frac{\rho k}{\omega} \quad (\text{H-62})$$

The production terms are approximated by

$$P_k = \mu_T \Omega^2 \quad (\text{H-63})$$

$$P_\omega = \gamma \rho \Omega^2 \quad (\text{H-64})$$

The constants are

$$\begin{aligned} \gamma &= \frac{\beta}{C_\mu} - \frac{\kappa^2}{\sigma_\omega \sqrt{C_\mu}} & \beta' &= C_\mu = 0.09 \\ \sigma_k &= 1/0.5 & \beta &= 0.075 \\ \sigma_\omega &= 1/0.5 & \kappa &= 0.41 \end{aligned} \quad (\text{H-65})$$

The minimum distance function d is not required by this model. However, at the present time it is still computed and stored by CFL3D. For the general form in Equation (H-30),

$$X_k = k \quad (\text{H-66})$$

$$S_{P,k} = \frac{1}{\rho} \mu_T \Omega^2 \left(\frac{M_\infty}{Re} \right) \quad (\text{H-67})$$

$$S_{D,k} = -\beta' k \omega \left(\frac{Re}{M_\infty} \right) \quad (\text{H-68})$$

$$D_k = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_T}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] \left(\frac{M_\infty}{Re} \right) \quad (\text{H-69})$$

and

$$X_{\omega} = \omega \quad (\text{H-70})$$

$$S_{P, \omega} = \gamma \Omega^2 \left(\frac{M_{\infty}}{Re} \right) \quad (\text{H-71})$$

$$S_{D, \omega} = -\beta \omega^2 \left(\frac{Re}{M_{\infty}} \right) \quad (\text{H-72})$$

$$D_{\omega} = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_T}{\sigma_{\omega}} \right) \frac{\partial \omega}{\partial x_j} \right] \left(\frac{M_{\infty}}{Re} \right) \quad (\text{H-73})$$

H.4.4 Menter's k-Omega SST Model

(ivisc = 7)

Menter's k- ω model is assessed in reference 27. Note that there are two corrections to this reference. Equation (1) should be

$$\begin{aligned} \sigma \mu_t &= \mu + \sigma \mu_t \\ \sigma^* \mu_t &= \mu + \sigma^* \mu_t \end{aligned} \quad (\text{H-74})$$

and Equation (17) should be

$$\Gamma = \max(2\Gamma_3, \Gamma_1) \quad (\text{H-75})$$

The two equations for this model are

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{1}{\rho} P_k \left(\frac{M_{\infty}}{Re} \right) - \beta' k \omega \left(\frac{Re}{M_{\infty}} \right) + \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_T}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] \left(\frac{M_{\infty}}{Re} \right) \quad (\text{H-76})$$

$$\begin{aligned} \frac{\partial \omega}{\partial t} + u_j \frac{\partial \omega}{\partial x_j} &= \frac{1}{\rho} P_{\omega} \left(\frac{M_{\infty}}{Re} \right) - \beta \omega^2 \left(\frac{Re}{M_{\infty}} \right) + \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_T}{\sigma_{\omega}} \right) \frac{\partial \omega}{\partial x_j} \right] \left(\frac{M_{\infty}}{Re} \right) \\ &\quad + 2(1 - F_1) \frac{1}{\sigma_{\omega_2}} \frac{1}{\omega} \frac{\partial k}{\partial x_j} \frac{\partial \omega}{\partial x_j} \left(\frac{M_{\infty}}{Re} \right) \end{aligned} \quad (\text{H-77})$$

In this model, C_{μ} is incorporated into the definition of ω . The eddy viscosity is given by

$$\mu_T = \min \left[\frac{\rho k}{\omega}, \frac{a_1 \rho k}{\Omega F_2} \left(\frac{Re}{M} \right) \right] \quad (\text{H-78})$$

The production terms are approximated by

$$P_k = \mu_T \Omega^2 \quad (\text{H-79})$$

$$P_\omega = \gamma \rho \Omega^2 \quad (\text{H-80})$$

The constants are calculated from $\phi = F_1 \phi_1 + (1 - F_1) \phi_2$, where the ϕ 's are the constants:

$$\begin{aligned} \gamma_1 &= \frac{\beta_1}{C_\mu} - \frac{\kappa^2}{\sigma_{\omega_1} \sqrt{C_\mu}} & \gamma_2 &= \frac{\beta_2}{C_\mu} - \frac{\kappa^2}{\sigma_{\omega_2} \sqrt{C_\mu}} \\ \sigma_{k_1} &= 1/0.85 & \sigma_{k_2} &= 1.0 \\ \sigma_{\omega_1} &= 1/0.5 & \sigma_{\omega_2} &= 1/0.856 \\ \beta_1 &= 0.075 & \beta_2 &= 0.0828 \\ \kappa &= 0.41 & a_1 &= 0.31 \\ \beta' &= C_\mu = 0.09 \end{aligned} \quad (\text{H-81})$$

$$\begin{aligned} F_1 &= \tanh(\Gamma^4) \\ \Gamma &= \min[\max(\Gamma_1, \Gamma_3), \Gamma_2] \\ \Gamma_1 &= \frac{500\nu}{d^2\omega} \left(\frac{M_\infty}{Re}\right)^2 & \Gamma_2 &= \frac{4\rho k}{d^2\sigma_{\omega_2}(CD_{k-\omega})} & \Gamma_3 &= \frac{\sqrt{k}}{C_\mu \omega d} \left(\frac{M_\infty}{Re}\right) \\ CD_{k-\omega} &= \max\left(\rho \frac{2}{\sigma_{\omega_2} \omega} \frac{\partial k}{\partial x_j} \frac{\partial \omega}{\partial x_j}, 1 \times 10^{-20}\right) \\ F_2 &= \tanh(\Pi^2) \\ \Pi &= \max(2\Gamma_3, \Gamma_1) \end{aligned} \quad (\text{H-82})$$

For the general form in Equation (H-30),

$$X_k = k \quad (\text{H-83})$$

$$S_{P,k} = \frac{1}{\rho} \mu_T \Omega^2 \left(\frac{M_\infty}{Re}\right) \quad (\text{H-84})$$

$$S_{D,k} = -\beta' k \omega \left(\frac{Re}{M_\infty}\right) \quad (\text{H-85})$$

$$D_k = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_T}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] \left(\frac{M_\infty}{Re}\right) \quad (\text{H-86})$$

and

$$X_{\omega} = \omega \quad (\text{H-87})$$

$$S_{P, \omega} = \gamma \Omega^2 \left(\frac{M_{\infty}}{Re} \right) \quad (\text{H-88})$$

$$S_{D, \omega} = -\beta \omega^2 \left(\frac{Re}{M_{\infty}} \right) + 2(1 - F_1) \sigma_{\omega_2} \frac{1}{\omega} \frac{\partial k}{\partial x_j} \frac{\partial \omega}{\partial x_j} \left(\frac{M_{\infty}}{Re} \right) \quad (\text{H-89})$$

$$D_{\omega} = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_T}{\sigma_{\omega}} \right) \frac{\partial \omega}{\partial x_j} \right] \left(\frac{M_{\infty}}{Re} \right) \quad (\text{H-90})$$

H.4.5 Abid k-Epsilon Model

(ivisc = 10)

For the Abid² k-ε model, the equations are

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{1}{\rho} P_k \left(\frac{M_{\infty}}{Re} \right) - \epsilon \left(\frac{Re}{M_{\infty}} \right) + \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_T}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] \left(\frac{M_{\infty}}{Re} \right) \quad (\text{H-91})$$

$$\frac{\partial \epsilon}{\partial t} + u_j \frac{\partial \epsilon}{\partial x_j} = \frac{1}{\rho} P_{\epsilon} \left(\frac{M_{\infty}}{Re} \right) - C_{\epsilon_2} \frac{\epsilon^2}{k} f_2 \left(\frac{Re}{M_{\infty}} \right) + \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_T}{\sigma_{\epsilon}} \right) \frac{\partial \epsilon}{\partial x_j} \right] \left(\frac{M_{\infty}}{Re} \right) \quad (\text{H-92})$$

$$\mu_T = C_{\mu} f_{\mu} \frac{\rho k^2}{\epsilon} \quad (\text{H-93})$$

The production terms are approximated by

$$P_k = \mu_T \Omega^2$$

$$P_{\epsilon} = C_{\epsilon_1} \frac{\epsilon}{k} \mu_T \Omega^2 \quad (\text{H-94})$$

The constants are

$$\begin{aligned} C_{\epsilon_1} &= 1.45 & \sigma_k &= 1.0 \\ C_{\epsilon_2} &= 1.83 & \sigma_{\epsilon} &= 1.4 \\ C_{\mu} &= 0.09 \end{aligned} \quad (\text{H-95})$$

The damping functions are given by

$$f_\mu = [1 + 4(Re_\tau^{-0.75})] \tanh(0.008 Re_k)$$

$$f_2 = \left[1 - \exp\left(-\frac{Re_k}{12}\right)\right]$$
(H-96)

where

$$Re_\tau = \frac{\rho k^2}{\mu \varepsilon}$$

$$Re_k = \frac{\rho \sqrt{k} d}{\mu} \left(\frac{Re}{M}\right)$$
(H-97)

where d is the distance to the nearest wall. For the general form in Equation (H-30),

$$X_k = k$$
(H-98)

$$S_{P,k} = \frac{1}{\rho} \mu_T \Omega^2 \left(\frac{M_\infty}{Re}\right)$$
(H-99)

$$S_{D,k} = -\varepsilon \left(\frac{Re}{M_\infty}\right)$$
(H-100)

$$D_k = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_T}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] \left(\frac{M_\infty}{Re}\right)$$
(H-101)

and

$$X_\varepsilon = \varepsilon$$
(H-102)

$$S_{P,\varepsilon} = \frac{1}{\rho} C_{\varepsilon_1} \frac{\varepsilon}{k} \mu_T \Omega^2 \left(\frac{M_\infty}{Re}\right)$$
(H-103)

$$S_{D,\varepsilon} = -C_{\varepsilon_2} \frac{\varepsilon^2}{k} f_2 \left(\frac{Re}{M_\infty}\right)$$
(H-104)

$$D_\varepsilon = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_T}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon}{\partial x_j} \right] \left(\frac{M_\infty}{Re}\right)$$
(H-105)

H.4.6 EASM Gatski-Speziale k -Omega Model

(**ivisc** = 8 and 12)

EASM stands for Explicit Algebraic Stress Model.³ **Ivisc** = 8 is the “linear” $k - \omega$ version, treated as an eddy viscosity model, whereas **ivisc** = 12 is the fully nonlinear $k - \omega$ version. The two versions obtain μ_T with identical methods. However, the nonlinear model includes nonlinear terms added to the Navier-Stokes equations, whereas the linear model does not. (Its effect is felt only through the μ_T term.)

The $k - \omega$ equations are of the same form as in the Wilcox $k - \omega$ model (Equation (H-60) and Equation (H-61)). However, C_μ is not incorporated into the definition of ω (it is now a variable coefficient), so,

$$\mu_T = C_\mu \rho \frac{k}{\omega} \quad (\text{H-106})$$

The constants are different as well:

$$\gamma = \beta - \frac{\kappa^2}{\sigma_\omega \sqrt{C_\mu^*}} \quad \begin{aligned} \sigma_k &= 1.4 \\ \sigma_\omega &= 2.2 \\ \beta' &= 1.0 \\ \beta &= 0.83 \\ \kappa &= 0.41 \end{aligned} \quad (\text{H-107})$$

The production terms are

$$\begin{aligned} P_k &= \tau_{ij}^T \frac{\partial u_i}{\partial x_j} \\ P_\omega &= \gamma \frac{\omega}{k} \tau_{ij}^T \frac{\partial u_i}{\partial x_j} \end{aligned} \quad (\text{H-108})$$

where τ_{ij}^T is given (dimensionally) in Equation (H-15). The variable C_μ and the coefficients K_1 and K_2 in Equation (H-15) are determined as follows:

$$C_\mu = \frac{3(1 + \eta^2) + 0.2(\eta^6 + \zeta^6)}{3 + \eta^2 + 6\zeta^2\eta^2 + 6\zeta^2 + \eta^6 + \zeta^6} \alpha_1 \quad (\text{H-109})$$

$$\eta = \frac{\alpha_2}{\omega} (S_{ij} S_{ij})^{\frac{1}{2}} \left(\frac{M_\infty}{Re} \right) \quad (\text{H-110})$$

$$\zeta = \frac{\alpha_3}{\omega} (W_{ij} W_{ij})^{\frac{1}{2}} \left(\frac{M_\infty}{Re} \right) \quad (\text{H-111})$$

$$\begin{aligned} K_1 &= \alpha_3 \\ K_2 &= 2\alpha_2 \end{aligned} \tag{H-112}$$

where

$$\begin{aligned} \alpha_1 &= \left(\frac{4}{3} - C_2\right) \frac{g}{2} \\ \alpha_2 &= (2 - C_3) \frac{g}{2} \\ \alpha_3 &= (2 - C_4) \frac{g}{2} \\ g &= \left(\frac{C_1}{2} + C_5 - 1\right)^{-1} \end{aligned} \tag{H-113}$$

(Note that η , ζ , and α are just used here to denote terms in the equations and do *not* reflect the definitions in the Nomenclature list.) Currently, the pressure-strain correlation is modeled with the Speziale-Sarkar-Gatski (SSG) correlation, which uses:

$$\begin{aligned} C_1 &= 6.8 & C_3 &= 1.25 & C_5 &= 1.88 \\ C_2 &= 0.36 & C_4 &= 0.4 & C_\mu^* &= 0.081 \end{aligned} \tag{H-114}$$

To improve convergence, the μ_T terms in τ_{ij}^T multiplying the nonlinear terms

$$S_{ik}W_{kj} - W_{ik}S_{kj} \text{ and } S_{ik}S_{kj} - \frac{1}{3}S_{kl}S_{lk}\delta_{ij}$$

are replaced by

$$\mu_T' = C_\mu' \rho \frac{k}{\omega} \tag{H-115}$$

where

$$C_\mu' = \frac{3(1 + \eta^2) + 0.2 \times 10^{-8}(\eta^6 + \zeta^6)}{3 + \eta^2 + 6\zeta^2\eta^2 + 6\zeta^2 + \eta^6 + \zeta^6} \alpha_1 \tag{H-116}$$

Also, the μ_T terms in the D_k and D_ω diffusion terms are replaced by

$$\mu_T^* = C_\mu^* \rho \frac{k}{\omega} \tag{H-117}$$

The minimum distance function d is not required by this model. However, at the present time it is still computed and stored by CFL3D. For the general form in Equation (H-30),

$$X_k = k \quad (\text{H-118})$$

$$S_{P,k} = \frac{1}{\rho} \tau_{ij}^T \frac{\partial u_i}{\partial x_j} \left(\frac{M_\infty}{Re} \right) \quad (\text{H-119})$$

$$S_{D,k} = -\beta' k \omega \left(\frac{Re}{M_\infty} \right) \quad (\text{H-120})$$

$$D_k = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_T^*}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] \left(\frac{M_\infty}{Re} \right) \quad (\text{H-121})$$

and

$$X_\omega = \omega \quad (\text{H-122})$$

$$S_{P,\omega} = \frac{1}{\rho} \gamma \frac{\omega}{k} \tau_{ij}^T \frac{\partial u_i}{\partial x_j} \left(\frac{M_\infty}{Re} \right) \quad (\text{H-123})$$

$$S_{D,\omega} = -\beta \omega^2 \left(\frac{Re}{M_\infty} \right) \quad (\text{H-124})$$

$$D_\omega = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_T^*}{\sigma_\omega} \right) \frac{\partial \omega}{\partial x_j} \right] \left(\frac{M_\infty}{Re} \right) \quad (\text{H-125})$$

H.4.7 EASM Gatski-Speziale k-Epsilon Model

(ivisc = 11)

The $k - \varepsilon$ version of the Gatski-Speziale EASM model³ is only implemented as a non-linear model. Its equations are identical to those of the Abid $k - \varepsilon$ model (Equation (H-91) and Equation (H-92)), with

$$\mu_T = C_\mu \frac{\rho k^2}{\varepsilon} \quad (\text{H-126})$$

(There is no f_μ term.) The constants are

$$\begin{aligned}
 C_{\varepsilon_1} &= C_{\varepsilon_2} - \frac{\kappa^2}{\sigma_\varepsilon \sqrt{C_\mu}} & C_{\varepsilon_2} &= 1.83 \\
 & & \sigma_k &= 1.0 \\
 & & \sigma_\varepsilon &= 1.3 \\
 & & \kappa &= 0.41
 \end{aligned} \tag{H-127}$$

The production terms are

$$\begin{aligned}
 P_k &= \tau_{ij}^T \frac{\partial u_i}{\partial x_j} \\
 P_\varepsilon &= C_{\varepsilon_1} \frac{\varepsilon}{k} \tau_{ij}^T \frac{\partial u_i}{\partial x_j}
 \end{aligned} \tag{H-128}$$

The variable C_μ , and all constants are identical to those given for the EASM $k - \omega$ model in “EASM Gatski-Speziale k -Omega Model” on page 286, except that ω is replaced by ε/k . The damping functions are given by

$$f_2 = \left[1 - \exp\left(-\frac{Re_k}{12}\right) \right] \tag{H-129}$$

where

$$Re_k = \frac{\rho \sqrt{k} d}{\mu} \left(\frac{Re}{M_\infty} \right) \tag{H-130}$$

The μ_T terms in τ_{ij}^T multiplying the nonlinear terms are replaced in exactly the same way as for the EASM $k - \omega$ model. Also, the μ_T terms in the D_k and D_ε diffusion terms are replaced by

$$\mu_T^* = C_\mu^* \frac{\rho k^2}{\varepsilon} \tag{H-131}$$

For the general form in Equation (H-30),

$$X_k = k \tag{H-132}$$

$$S_{P,k} = \frac{1}{\rho} \tau_{ij}^T \frac{\partial u_i}{\partial x_j} \left(\frac{M_\infty}{Re} \right) \tag{H-133}$$

$$S_{D,k} = -\varepsilon \left(\frac{Re}{M_\infty} \right) \tag{H-134}$$

$$D_k = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_T^*}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] \left(\frac{M_\infty}{Re} \right) \quad (\text{H-135})$$

and

$$X_\varepsilon = \varepsilon \quad (\text{H-136})$$

$$S_{P,\varepsilon} = \frac{1}{\rho} C_{\varepsilon_1} \frac{\varepsilon}{k} \tau_{ij}^T \frac{\partial u_i}{\partial x_j} \left(\frac{M_\infty}{Re} \right) \quad (\text{H-137})$$

$$S_{D,\varepsilon} = -C_{\varepsilon_2} \frac{\varepsilon^2}{k} f_2 \left(\frac{Re}{M_\infty} \right) \quad (\text{H-138})$$

$$D_\varepsilon = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_T^*}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon}{\partial x_j} \right] \left(\frac{M_\infty}{Re} \right) \quad (\text{H-139})$$

H.4.8 EASM Girimaji k-Epsilon Model

(**ivisc** = 9 and 13)

The Girimaji²¹ version of the EASM k – ε model is implemented both as a “linear” (eddy-viscosity) (**ivisc** = 9) and “nonlinear” (**ivisc** = 13) version. The k – ε equations are

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{1}{\rho} P_k \left(\frac{M_\infty}{Re} \right) - \varepsilon \left(\frac{Re}{M_\infty} \right) + \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_T}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] \left(\frac{M_\infty}{Re} \right) \quad (\text{H-140})$$

$$\begin{aligned} \frac{\partial \varepsilon}{\partial t} + u_j \frac{\partial \varepsilon}{\partial x_j} = & \frac{1}{\rho} P_\varepsilon \left(\frac{M_\infty}{Re} \right) - C_{\varepsilon_2} \frac{\varepsilon^2}{k} \left(\frac{Re}{M_\infty} \right) + \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_T}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon}{\partial x_j} \right] \left(\frac{M_\infty}{Re} \right) \\ & + C_{\varepsilon_2} \frac{\varepsilon}{k} \frac{2\mu}{\rho} \left(\frac{\partial \sqrt{k}}{\partial x_j} \right)^2 \left(\frac{M_\infty}{Re} \right) \end{aligned} \quad (\text{H-141})$$

$$\mu_T = -G_1 f_\mu \frac{\rho k^2}{\varepsilon} \quad (\text{H-142})$$

The extra $(\partial \sqrt{k} / \partial x_j)^2$ term in the ε equation was added to replace the f_2 damping function term on the ε^2/k term, which is needed because $\varepsilon^2/k \rightarrow \infty$ at the wall. The constants are

$$\begin{aligned}
C_{\varepsilon_1} &= 1.44 & \sigma_k &= 1.0 \\
C_{\varepsilon_2} &= 1.83 & \sigma_\varepsilon &= 1.3
\end{aligned}
\tag{H-143}$$

The production terms are approximated by

$$\begin{aligned}
P_k &= \tau_{ij}^T \frac{\partial u_i}{\partial x_j} \\
P_\varepsilon &= C_{\varepsilon_1 k} \frac{\varepsilon}{k} \tau_{ij}^T \frac{\partial u_i}{\partial x_j}
\end{aligned}
\tag{H-144}$$

The damping functions f_μ is

$$f_\mu = \tanh \left[0.015 \frac{\sqrt{k} d \rho}{\mu} \left(\frac{Re}{M_\infty} \right) \right] \tag{H-145}$$

τ_{ij}^T is given (dimensionally) in Equation (H-15). However, K_1 and K_2 in that equation are now given by

$$\begin{aligned}
K_1 &= \frac{G_2}{G_1} \\
K_2 &= -\frac{G_3}{G_1}
\end{aligned}
\tag{H-146}$$

and

$$\begin{aligned}
\text{for } \eta_1 &= 0 & G_1 &= \frac{L_1^0 L_2}{(L_1^0)^2 + 2\eta_2 (L_4)^2} \\
\text{for } L_1^1 &= 0 & G_1 &= \frac{L_1^0 L_2}{(L_1^0)^2 + \frac{2}{3}\eta_1 (L_3)^2 + 2\eta_2 (L_4)^2} \\
\text{for } D > 0 & & G_1 &= -\frac{p}{3} + \left(-\frac{b}{2} + \sqrt{D} \right)^{\frac{1}{3}} + \left(-\frac{b}{2} - \sqrt{D} \right)^{\frac{1}{3}} \\
\text{for } D < 0, b < 0 & & G_1 &= -\frac{p}{3} + 2\sqrt{-\frac{a}{3}} \cos\left(\frac{\theta}{3}\right) \\
\text{for } D < 0, b > 0 & & G_1 &= -\frac{p}{3} + 2\sqrt{-\frac{a}{3}} \cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right)
\end{aligned}
\tag{H-147}$$

where

$$\eta_1 = \left(\frac{k}{\epsilon}\right)^2 S_{ij} S_{ij} \left(\frac{M_\infty}{Re}\right)^2 \quad (\text{H-148})$$

$$\eta_2 = \left(\frac{k}{\epsilon}\right)^2 W_{ij} W_{ij} \left(\frac{M_\infty}{Re}\right)^2$$

$$p = -\frac{2L_1^0}{\eta_1 L_1^1} \quad r = -\frac{L_1^0 L_2}{(\eta_1 L_1^1)^2} \quad (\text{H-149})$$

$$q = \frac{1}{(\eta_1 L_1^1)^2} \left[(L_1^0)^2 + \eta_1 L_1^1 L_2 - \frac{2}{3} \eta_1 (L_3)^2 + 2\eta_2 (L_4)^2 \right] \quad (\text{H-150})$$

$$a = \left(q - \frac{p^2}{3}\right) \quad b = \frac{1}{27}(2p^3 - 9pq + 27r) \quad (\text{H-151})$$

$$D = \frac{b^2}{4} + \frac{a^3}{27} \quad \cos\theta = \frac{-b/2}{\sqrt{-a^3/27}}$$

$$G_2 = \frac{-L_4 G_1}{L_1^0 - \eta_1 L_1^1 G_1} \quad G_3 = \frac{+2L_3 G_1}{L_1^0 - \eta_1 L_1^1 G_1} \quad (\text{H-152})$$

$$L_1^0 = \frac{C_1^0}{2} - 1 \quad L_2 = \frac{C_2}{2} - \frac{2}{3} \quad (\text{H-153})$$

$$L_1^1 = C_1^1 + 2 \quad L_3 = \frac{C_3}{2} - 1$$

$$L_4 = \frac{C_4}{2} - 1$$

$$C_1^0 = 3.4 \quad C_2 = 0.36$$

$$C_1^1 = 1.8 \quad C_3 = 1.25 \quad (\text{H-154})$$

$$C_4 = 0.4$$

For the general form in Equation (H-30),

$$X_k = k \quad (\text{H-155})$$

$$S_{P,k} = \frac{1}{\rho} \tau_{ij} \frac{\partial u_i}{\partial x_j} \left(\frac{M_\infty}{Re}\right) \quad (\text{H-156})$$

$$S_{D,k} = -\varepsilon \left(\frac{Re}{M_\infty} \right) \quad (\text{H-157})$$

$$D_k = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_T}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] \left(\frac{M_\infty}{Re} \right) \quad (\text{H-158})$$

and

$$X_\varepsilon = \varepsilon \quad (\text{H-159})$$

$$S_{P,\varepsilon} = \frac{1}{\rho} C_{\varepsilon_1} \frac{\varepsilon}{k} \tau_{ij}^T \frac{\partial u_i}{\partial x_j} \left(\frac{M_\infty}{Re} \right) + C_{\varepsilon_2} \frac{\varepsilon}{k} \frac{2\mu}{\rho} \left(\frac{\partial \sqrt{k}}{\partial x_j} \right)^2 \left(\frac{M_\infty}{Re} \right) \quad (\text{H-160})$$

$$S_{D,\varepsilon} = -C_{\varepsilon_2} \frac{\varepsilon^2}{k} \left(\frac{Re}{M_\infty} \right) \quad (\text{H-161})$$

$$D_\varepsilon = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_T}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon}{\partial x_j} \right] \left(\frac{M_\infty}{Re} \right) \quad (\text{H-162})$$

H.5 Generalized Coordinate Form

For the transformation from Cartesian coordinates to generalized coordinates, for example,

$$u_j \frac{\partial k}{\partial x_j} = U \frac{\partial k}{\partial \xi} + V \frac{\partial k}{\partial \eta} + W \frac{\partial k}{\partial \zeta} \quad (\text{H-163})$$

where

$$\begin{aligned} U &= \xi_x u + \xi_y v + \xi_z w + \xi_t \\ V &= \eta_x u + \eta_y v + \eta_z w + \eta_t \\ W &= \zeta_x u + \zeta_y v + \zeta_z w + \zeta_t \end{aligned} \quad (\text{H-164})$$

In the diffusion-type terms, cross-derivative terms are neglected. For example,

$$\begin{aligned}
\frac{\partial}{\partial x_j} \left(\mu \frac{\partial k}{\partial x_j} \right) &\equiv \xi_x \frac{\partial}{\partial \xi} \left(\xi_x \mu \frac{\partial k}{\partial \xi} \right) + \xi_y \frac{\partial}{\partial \xi} \left(\xi_y \mu \frac{\partial k}{\partial \xi} \right) + \xi_z \frac{\partial}{\partial \xi} \left(\xi_z \mu \frac{\partial k}{\partial \xi} \right) \\
&+ \eta_x \frac{\partial}{\partial \eta} \left(\eta_x \mu \frac{\partial k}{\partial \eta} \right) + \eta_y \frac{\partial}{\partial \eta} \left(\eta_y \mu \frac{\partial k}{\partial \eta} \right) + \eta_z \frac{\partial}{\partial \eta} \left(\eta_z \mu \frac{\partial k}{\partial \eta} \right) \\
&+ \zeta_x \frac{\partial}{\partial \zeta} \left(\zeta_x \mu \frac{\partial k}{\partial \zeta} \right) + \zeta_y \frac{\partial}{\partial \zeta} \left(\zeta_y \mu \frac{\partial k}{\partial \zeta} \right) + \zeta_z \frac{\partial}{\partial \zeta} \left(\zeta_z \mu \frac{\partial k}{\partial \zeta} \right)
\end{aligned} \tag{H-165}$$

An example of the discretization is given for the first term on the right-hand side of Equation (H-165):

$$\xi_x \frac{\partial}{\partial \xi} \left(\xi_x \mu \frac{\partial k}{\partial \xi} \right) \rightarrow \xi_{x_i} \xi_{x_{i+\frac{1}{2}}} \mu_{i+\frac{1}{2}} \left(\frac{\partial k}{\partial \xi} \right)_{i+\frac{1}{2}} - \xi_{x_i} \xi_{x_{i-\frac{1}{2}}} \mu_{i-\frac{1}{2}} \left(\frac{\partial k}{\partial \xi} \right)_{i-\frac{1}{2}} \tag{H-166}$$

where

$$\left(\frac{\partial k}{\partial \xi} \right)_{i+\frac{1}{2}} = k_{i+1} - k_i \quad \left(\frac{\partial k}{\partial \xi} \right)_{i-\frac{1}{2}} = k_i - k_{i-1} \tag{H-167}$$

The locations of the indices are shown in Figure H-2.

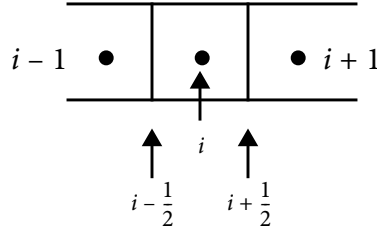


Figure H-2. Definition of left and right states for generalized coordinate transformation.

H.6 Initial Conditions

For the Baldwin-Barth and Spalart-Allmaras models, the levels of the turbulence quantities R and \hat{v} are initialized at their free-stream levels everywhere in the flow field. For the two-equation models, it helps the solution start-up phase to initialize the k and ω (or k and ε) values to crude profiles that simulate typical profiles in a boundary layer. Currently, for $k - \omega$:

$$\begin{aligned}
k_{IC} &= \max(k_{\infty}, -C_1 d^2 + C_2 d) \\
\omega_{IC} &= \max\left(-12444d + 0.54, \frac{C_{\mu} k_{IC}}{v3d}\right)
\end{aligned} \tag{H-168}$$

where d is the distance to the nearest wall, scaled by $Re/5.e6$; and, for EASM models, $C_{\mu} = 0.09$; otherwise, $C_{\mu} = 1.0$. Also,

$$\begin{aligned}
v3d &= \max\left[\frac{100k_{IC}}{t_{\max}}, 0.009\right] \\
t_{\max} &= -C_1 S_{\max}^2 + C_2 S_{\max} \\
S_{\max} &= \frac{C_2}{2C_1} \\
C_1 &= 45.8 \quad C_2 = 1.68
\end{aligned} \tag{H-169}$$

For $k - \varepsilon$:

$$k_{IC} = \min\{zk4, \max[zk1, \min(zk2, zk3)]\} \tag{H-170}$$

$$\begin{aligned}
zk1 &= k_{\infty} \\
zk2 &= 10^{-471d + 0.47} \\
zk3 &= 10^{-37.5d - 3.7} \\
zk4 &= 6.7d
\end{aligned} \tag{H-171}$$

$$\varepsilon_{IC} = \min\{ep4, \max[ep1, \min(ep2, ep3)]\} \tag{H-172}$$

$$\begin{aligned}
ep1 &= \varepsilon_{\infty} \\
ep2 &= 10^{-555d - 6} \\
ep3 &= 10^{-280d - 9.2} \\
ep4 &= \min(1 \times 10^{20}, 10^{13333d - 9.8})
\end{aligned} \tag{H-173}$$

H.7 Boundary Conditions

H.7.1 Free-stream Levels

For the Baldwin-Barth model,

$$R_{\infty} = 0.1 \quad (\text{H-174})$$

For the Spalart-Allmaras model,

$$\hat{v}_{\infty} = 1.341946 \quad (\text{H-175})$$

For the standard $k - \omega$ or Menter's SST models (C_{μ} is incorporated into the definition of ω),

$$\begin{aligned} k_{\infty} &= 9 \times 10^{-9} \\ \omega_{\infty} &= 1 \times 10^{-6} \end{aligned} \quad (\text{H-176})$$

For the EASM ($k - \omega$) models,

$$\begin{aligned} k_{\infty} &= 9 \times 10^{-9} \\ \omega_{\infty} &= 9 \times 10^{-8} \end{aligned} \quad (\text{H-177})$$

For the $k - \epsilon$ models,

$$\begin{aligned} k_{\infty} &= 1 \times 10^{-9} \\ \epsilon_{\infty} &= 1 \times 10^{-17} \end{aligned} \quad (\text{H-178})$$

The end result for all models is

$$\mu_{T, \infty} \cong 0.009 \quad (\text{H-179})$$

in the free stream. Note that $\omega_{\infty} = 9 \times 10^{-8}$ and $\epsilon_{\infty} = 1 \times 10^{-17}$ for the EASM models are based on $C_{\mu} = 0.09$, which is only approximate since C_{μ} is variable. Also note that for two-equation models in the code, ϵ (or ω) is represented by the 1st turbulence variable (e.g., **tur10**), and k by the 2nd (e.g., **tur20**).

H.7.2 Boundary Conditions at Solid Walls

With the parameters known at the cell-centers near the solid wall boundary (see the wall region schematic in Figure H-3), the following boundary conditions are applied.

For Baldwin-Barth,

$$R_w = 0 \quad (\text{H-180})$$

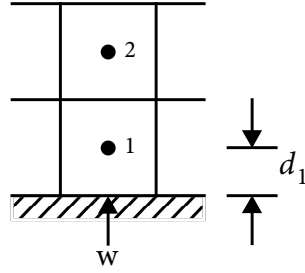


Figure H-3. Parameter locations at wall boundary.

For Spalart-Allmaras,

$$\hat{v} = 0 \quad (\text{H-181})$$

For the $k - \omega$ models,

$$k_w = 0 \quad (\text{H-182})$$

$$\omega_w = \frac{60\mu_1}{\rho_1\beta(d_1)^2} \left(\frac{M_\infty}{Re} \right)^2 \quad (\text{H-183})$$

where

$\beta = 0.83$ for the EASM $k - \omega$ models and

$\beta = 0.075$ otherwise.

For the $k - \epsilon$ models,

$$k_w = 0 \quad (\text{H-184})$$

$$\epsilon_w = \frac{2\mu_1}{\rho_1} \left(\frac{\partial \sqrt{k}}{\partial n} \right)_w^2 \left(\frac{M_\infty}{Re} \right)^2 \quad (\text{H-185})$$

where n is the direction normal to the wall.

Note that the actual wall boundary conditions for the turbulence quantities (unlike the primitive variables) are applied at ghost cells. Hence,

$$\chi_{BC} = 2\chi_w - \chi_1 \quad (\text{H-186})$$

where χ represents R , \hat{v} , k , ω , or ϵ .

Version 4.1 of CFL3D applied the above boundary condition for ω (Equation (H-183)) at the ghost cell center rather than at the wall. This is not correct, but supposedly any wall boundary condition for ω that is bigger by some factor than the “analytical” value $6\mu_1/(\rho\beta y^2)$ gives basically the same result. (See reference 26.) The current version of CFL3D (Version 5.0) obtains the ghost cell value using linear interpolation from ω_w and ω_1 (or ε_w and ε_1).

H.8 Solution Method

The one or two equations are solved, decoupled, using implicit approximate factorization (AF). The S_p terms are treated explicitly, lagged in time while the S_D and D terms are treated implicitly (they are linearized and a term is brought to the left-hand-side of the equations). This procedure is described by Spalart and Allmaras³⁵ (strategy #3). The advective terms are discretized using first-order upwinding. Treating the destruction terms implicitly helps increase the diagonal dominance of the left-hand-side matrix. Most of the S_D terms are linearized in a simplified fashion by assuming no coupling between the variables. For example,

$$\begin{aligned}\beta'k\omega^{(n+1)} &\equiv \beta'k\omega^{(n)} + \frac{\partial}{\partial k}(\beta'k\omega)\Delta k \\ &\equiv \beta'k\omega^{(n)} + \beta'\omega\Delta k\end{aligned}\tag{H-187}$$

For the S_D terms in the $k - \varepsilon$ equations, however, a more sophisticated approach was taken. When linearizing, the k and ε variables are assumed to be coupled, with the eddy viscosity μ_T assumed to be fixed. For example, in the k equation, $S_D = -\varepsilon(Re/M_\infty)$ is linearized as follows:

$$\begin{aligned}\varepsilon^{(n+1)} &\equiv \varepsilon^{(n)} + \frac{\partial}{\partial k}(\varepsilon)\Delta k \\ &\equiv \varepsilon^{(n)} + \frac{\partial}{\partial k}\left(\frac{C_\mu f_\mu \rho k^2}{\mu_T}\right)\Delta k \\ &\equiv \varepsilon^{(n)} + 2\frac{C_\mu f_\mu \rho k}{\mu_T}\Delta k \\ &\equiv \varepsilon^{(n)} + 2\frac{\varepsilon}{k}\Delta k\end{aligned}\tag{H-188}$$

The $S_D = -C_{\varepsilon_2} \frac{\varepsilon^2}{k} f_2(Re/M_\infty)$ term in the ε equation is treated similarly, for consistency:

$$\begin{aligned}
 \frac{\varepsilon^{2(n+1)}}{k} &\cong \frac{\varepsilon^{2(n)}}{k} + \frac{\partial}{\partial \varepsilon} \left(\frac{\varepsilon^2}{k} \right) \Delta \varepsilon \\
 &\cong \frac{\varepsilon^{2(n)}}{k} + \frac{\partial}{\partial \varepsilon} \left[\left(\frac{C_\mu f_\mu \rho}{\mu_T} \right)^{\frac{1}{2}} \varepsilon^{\frac{3}{2}} \right] \Delta \varepsilon \\
 &\cong \frac{\varepsilon^{2(n)}}{k} + \frac{3}{2} \left(\frac{C_\mu f_\mu \rho}{\mu_T} \right)^{\frac{1}{2}} \Delta \varepsilon \\
 &\cong \frac{\varepsilon^{2(n)}}{k} + \frac{3\varepsilon}{2k} \Delta \varepsilon
 \end{aligned} \tag{H-189}$$

Menter's SST $k - \omega$ model has an additional cross-diffusion term in the ω equation. This term is treated implicitly (See reference 26).

$$\begin{aligned}
 C_\omega^{(n+1)} &\cong C_\omega^{(n)} + \frac{\partial}{\partial \omega} C_\omega \Delta \omega \\
 &\cong C_\omega^{(n)} - \frac{|C_\omega|}{\omega} \Delta \omega
 \end{aligned} \tag{H-190}$$

(The negative sign insures that, when taken to the left-hand side, this term increases the diagonal dominance of the implicit matrix.)

H.8.1 Example

This example shows the solution method for the k equation in Menter's SST model. All other equations are solved in a similar fashion.

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{1}{\rho} P_k \left(\frac{M_\infty}{Re} \right) - \beta' k \omega \left(\frac{Re}{M_\infty} \right) + \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_T}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] \left(\frac{M_\infty}{Re} \right) \tag{H-191}$$

Written in generalized coordinates, with the production term treated explicitly (lagged in time), this equation becomes:

$$\begin{aligned}
\frac{\partial k^{(n+1)}}{\partial t} &= -U \frac{\partial k^{(n+1)}}{\partial \xi} - V \frac{\partial k^{(n+1)}}{\partial \eta} - W \frac{\partial k^{(n+1)}}{\partial \zeta} \\
&+ \frac{1}{\rho} P_k \left(\frac{M_\infty}{Re} \right)^{(n)} - \beta' k \omega \left(\frac{Re}{M_\infty} \right)^{(n+1)} \\
&+ \frac{1}{\rho} \left[\frac{\partial}{\partial \xi} \left\{ (\xi_x^2 + \xi_y^2 + \xi_z^2) \left(\mu + \frac{\mu_T}{\sigma_k} \right) \frac{\partial k}{\partial \xi} \right\} \right. \\
&+ \frac{\partial}{\partial \eta} \left\{ (\eta_x^2 + \eta_y^2 + \eta_z^2) \left(\mu + \frac{\mu_T}{\sigma_k} \right) \frac{\partial k}{\partial \eta} \right\} \\
&+ \left. \frac{\partial}{\partial \zeta} \left\{ (\zeta_x^2 + \zeta_y^2 + \zeta_z^2) \left(\mu + \frac{\mu_T}{\sigma_k} \right) \frac{\partial k}{\partial \zeta} \right\} \right] \left(\frac{M_\infty}{Re} \right)^{(n+1)} \\
&= \text{RHS}_k
\end{aligned} \tag{H-192}$$

where the superscripts denote the time level. Note that the terms $(\xi_x^2 + \xi_y^2 + \xi_z^2)$ etc. are not strictly correct as written. This is really short-hand notation. The correct way to expand the diffusion terms is given in “Generalized Coordinate Form” on page 294. This section also indicates how these metric terms are discretized.

$$\begin{aligned}
U &= \xi_x u + \xi_y v + \xi_z w + \xi_t \\
V &= \eta_x u + \eta_y v + \eta_z w + \eta_t \\
W &= \zeta_x u + \zeta_y v + \zeta_z w + \zeta_t
\end{aligned} \tag{H-193}$$

Define

$$\Delta k = k^{(m+1)} - k^{(m)} \tag{H-194}$$

Then, t-TS subiterations are used (see Appendix B). (If no subiterations, then $m = n$.)

$$\frac{(1 + \phi)(k^{(m+1)} - k^{(n)}) - \phi(k^{(n)} - k^{(n-1)})}{\Delta t} = \text{RHS}_k \tag{H-195}$$

$\phi = 0$ for 1st order in time (also used for non-time-accurate runs), $\phi = 1/2$ for 2nd order in time. Linearize the right-hand-side terms that are taken at time level $(m + 1)$:

$$-U \frac{\partial k^{(m+1)}}{\partial \xi} \cong -U \frac{\partial k^{(m)}}{\partial \xi} - U \delta_\xi^{upwind}(\Delta k) \tag{H-196}$$

$$-\left(\frac{Re}{M_\infty}\right)\beta'k\omega^{(m+1)} \equiv -\left(\frac{Re}{M_\infty}\right)\beta'k\omega^{(m)} - \left(\frac{Re}{M_\infty}\right)\beta'\omega(\Delta k) \quad (\text{H-197})$$

$$\left(\frac{M_\infty}{Re}\right)\frac{1}{\rho}\frac{\partial}{\partial\xi}\left\{\chi_\xi\frac{\partial k}{\partial\xi}\right\}^{(m+1)} \equiv \left(\frac{M_\infty}{Re}\right)\frac{1}{\rho}\frac{\partial}{\partial\xi}\left\{\chi_\xi\frac{\partial k}{\partial\xi}\right\}^{(m)} + \left(\frac{M_\infty}{Re}\right)\frac{1}{\rho}\frac{\partial}{\partial\xi}\{\chi_\xi\delta_\xi(\Delta k)\} \quad (\text{H-198})$$

where χ_ξ represents $(\xi_x^2 + \xi_y^2 + \xi_z^2)\left(\mu + \frac{\mu_T}{\sigma_k}\right)$.

The equation is approximately factored and the contribution of the S_D term to the left-hand side is added in the first factor only.

$$\begin{aligned} \frac{1}{(1+\phi)^2} & \left[(1+\phi)I + \Delta t U \delta_\xi^{upwind} + \Delta t \left(\frac{Re}{M_\infty}\right)\beta'\omega - \Delta t \left(\frac{M_\infty}{Re}\right)\frac{1}{\rho}\frac{\partial}{\partial\xi}(\chi_\xi\delta_\xi) \right] \\ & \left[(1+\phi)I + \Delta t V \delta_\eta^{upwind} - \Delta t \left(\frac{M_\infty}{Re}\right)\frac{1}{\rho}\frac{\partial}{\partial\eta}(\chi_\eta\delta_\eta) \right] \\ & \left[(1+\phi)I + \Delta t W \delta_\zeta^{upwind} - \Delta t \left(\frac{M_\infty}{Re}\right)\frac{1}{\rho}\frac{\partial}{\partial\zeta}(\chi_\zeta\delta_\zeta) \right] \Delta k = R \end{aligned} \quad (\text{H-199})$$

$$R = \Delta t \text{RHS}_k + \phi \Delta k^{(n-1)} - (1+\phi)(k^{(m)} - k^{(n)}) \quad (\text{H-200})$$

This equation is solved in a series of three sweeps in each of the three coordinate directions:

$$\begin{aligned} & \left[(1+\phi)I + \Delta t U \delta_\xi^{upwind} + \Delta t \left(\frac{Re}{M_\infty}\right)\beta'\omega - \Delta t \left(\frac{M_\infty}{Re}\right)\frac{1}{\rho}\frac{\partial}{\partial\xi}(\chi_\xi\delta_\xi) \right] \Delta k^* = R \\ & \left[(1+\phi)I + \Delta t V \delta_\eta^{upwind} - \Delta t \left(\frac{M_\infty}{Re}\right)\frac{1}{\rho}\frac{\partial}{\partial\eta}(\chi_\eta\delta_\eta) \right] \Delta k^{**} = (1+\phi)\Delta k^* \\ & \left[(1+\phi)I + \Delta t W \delta_\zeta^{upwind} - \Delta t \left(\frac{M_\infty}{Re}\right)\frac{1}{\rho}\frac{\partial}{\partial\zeta}(\chi_\zeta\delta_\zeta) \right] \Delta k = (1+\phi)\Delta k^{**} \end{aligned} \quad (\text{H-201})$$

$$k^{(m+1)} = k^{(m)} + \Delta k$$

Each sweep requires the solution of a scalar tridiagonal matrix. In versions of CFL3D prior to March 2002, $\phi = 0$ always for the turbulence equations, regardless of the temporal order of accuracy of the mean flow equations.

H.9 Limiting

As a precaution, many of the turbulence quantities are limited in practice. Whether all of these conditions are necessary or not for the models to work and/or converge well is

unknown. Some are probably included only because they were tried at some point and never removed.

- In all models, the linearized S_D term is added to the left-hand-side matrix only if its contribution is positive.
- In all models, the terms D , \hat{v} , k , ω , and ε are not allowed to become negative. If the equation “wants” to yield a negative value, it is instead set to some very small positive number. (A counter also sums the number of times this happen and it is output in the *.turses output file.)
- In all the $k - \omega$ models as well as the $k - \varepsilon$ models for Abid and Gatski-Speziale, the production term in the k equation (i.e.

$$\frac{1}{\rho} \tau_{ij}^T \frac{\partial u_i}{\partial x_j} \left(\frac{M_\infty}{Re} \right)$$

or an approximation thereof) is limited to be less than or equal to 20 times the destruction term in that equation.

- In all EASM models, the τ_{11} , τ_{22} , and τ_{33} terms are limited to be positive (realizability constraint).
- In all EASM models, the production term $\tau_{ij}^T \frac{\partial u_i}{\partial x_j}$ is limited to be positive.
- In the Girimaji EASM models, the production term

$$\frac{1}{\rho} \tau_{ij}^T \frac{\partial u_i}{\partial x_j} \left(\frac{M_\infty}{Re} \right)$$

is limited (in both the k and ε equations) to be less than or equal to $40\varepsilon \left(\frac{Re}{M_\infty} \right)$.

- In the Girimaji EASM models, the term

$$\varepsilon \left(\frac{Re}{M_\infty} \right) - 2 \frac{\mu}{\rho} \left(\frac{\partial \sqrt{k}}{\partial x_j} \right)^2 \left(\frac{M_\infty}{Re} \right)$$

is limited to be positive.

- In all two-equation models, the nondimensional eddy viscosity μ_T is limited to be less than or equal to 100,000.
- In the Gatski-Speziale EASM models, η^2 and ζ^2 are each limited to be less than or equal to 10.
- In the Gatski-Speziale EASM models, C_μ is limited to be between 0.005 and 0.187.
- In the $k - \varepsilon$ models, f_μ is limited to be less than or equal to 1.
- In the Girimaji EASM models, η_1 and η_2 are each limited to be less than or equal to 1200.
- In the Girimaji EASM model, G_1 is limited to be between -0.005 and -0.2.
- In the Gatski-Speziale EASM $k - \omega$ models, “ ω ” in the denominator of the nonlinear terms in τ_{ij}^T is limited to be greater than or equal to the free-stream value of ω .
- In the EASM $k - \varepsilon$ models, “ ε ” in the denominator of the nonlinear terms in τ_{ij}^T is limited to be greater than or equal to the free-stream value of ε *only* in terms that appear in the Navier-Stokes equations. This limiting is not done for the τ_{ij} terms that appear in the S_p terms in the k and ε equations.
- In the one- and two-equation models, the diffusion term in one coordinate direction, for example, can be written as

$$\frac{\partial}{\partial \xi} \left[B \frac{\partial \chi}{\partial \xi} \right] = B_{i+\frac{1}{2}} \chi_{i+1} - \left(B_{i+\frac{1}{2}} + B_{i-\frac{1}{2}} \right) \chi_i + B_{i-\frac{1}{2}} \chi_{i-1}$$

In Baldwin-Barth and Spalart-Allmaras, the terms are limited as follows:

$$B_{i+\frac{1}{2}} = \max \left(B_{i+\frac{1}{2}}, 0 \right)$$

$$B_{i-\frac{1}{2}} = \max \left(B_{i-\frac{1}{2}}, 0 \right)$$

H.10 Wall Function

A simple approach for wall functions is taken from Abdol-Hamid, Lakshmanan, and Carlson¹. This approach modifies the eddy-viscosity value $\mu_{T,w}$ at the wall only. The wall shear stress is estimated from law-of-the-wall values at the first cell center off the wall and is used to determine an “equivalent” eddy viscosity at the wall.

Wall functions can be employed when the grid spacing is too coarse in the direction normal to the wall; typically y^+ lies in the range $30 < y^+ < 200$. (The y^+ value of the first grid point off the wall *should* be $O(1)$ to ensure decent turbulent computations when no wall functions are used. See Section 2.2.) Wall functions are invoked in CFL3D by setting the value of `ivisc(m)` to be negative.

A caution: While wall functions can work very well, they can also sometimes cause problems. They are not strictly valid for separated flows, although many people use them anyway with reasonable success. It is recommended that wall functions *not* be used with the Baldwin-Lomax model in CFL3D. With other turbulence models, it is recommended that they be used sparingly, if at all. In our opinion, it is better to make a turbulent grid ($y^+ = O(1)$) whenever possible and *avoid the use of wall functions*.

The wall function replaces the eddy viscosity at the ghost cell center at walls as follows:

$$\begin{aligned}\mu_{T,ghost} &= 2\mu_{T,w} - \mu_{T,1} \\ \mu_{T,w} &= \mu_1 \left[\frac{n^{+2} \mu_1}{\rho_1 \left(\frac{\partial u}{\partial y} \right)_1 d_1^2} \left(\frac{M_\infty}{Re} \right) - 1 \right] \\ \left(\frac{\partial u}{\partial y} \right)_1 &= \frac{\sqrt{(u_1 - u_w)^2 + (v_1 - v_w)^2 + (w_1 - w_w)^2}}{d_1}\end{aligned}\tag{H-202}$$

where d_1 is the distance to the nearest wall from the first cell center off the wall and

$$\begin{aligned}
 &\text{for } R_c \leq 20.24 & n^+ &= \sqrt{R_c} \\
 &\text{for } 20.24 < R_c \leq 435 & n^+ &= a_0(1) + \sum_{n=2}^7 a_0(n) R_c^{n-1} \\
 &\text{for } 435 < R_c \leq 4000 & n^+ &= a_1(1) + \sum_{n=2}^5 a_1(n) R_c^{n-1} \\
 &\text{for } R_c > 4000 & n^+ &= a_2(1) + \sum_{n=2}^3 a_2(n) R_c^{n-1}
 \end{aligned} \tag{H-203}$$

$$R_c = \frac{\rho_1 \sqrt{(u_1 - u_w)^2 + (v_1 - v_w)^2 + (w_1 - w_w)^2} d_1}{\mu_1} \left(\frac{Re}{M_\infty} \right) \tag{H-204}$$

$$\begin{aligned}
 &a_1(1) = 5.777191 \\
 &a_0(1) = 2.354039 & a_1(2) &= 6.8756983 \times 10^{-2} \\
 &a_0(2) = 0.117984 & a_1(3) &= -7.1582745 \times 10^{-6} \\
 &a_0(3) = -4.2899192 \times 10^{-4} & a_1(4) &= 1.5594904 \times 10^{-9} \\
 &a_0(4) = 2.0404148 \times 10^{-6} & a_1(5) &= -1.4865778 \times 10^{-13} \\
 &a_0(5) = -5.1775775 \times 10^{-9} & & \\
 &a_0(6) = 6.2687308 \times 10^{-12} & a_2(1) &= 31.08654 \\
 &a_0(7) = -2.916958 \times 10^{-15} & a_2(2) &= 5.0429072 \times 10^{-2} \\
 & & a_2(3) &= -2.0072314 \times 10^{-8}
 \end{aligned} \tag{H-205}$$