APPENDIX A

Governing Equations

The computational algorithm employed in CFL3D for the three-dimensional Navier-Stokes code CFL3D is described in Thomas et al.[37] The governing equations, which are the thin-layer approximations to the three-dimensional time-dependent compressible Navier-Stokes equations, can be written in terms of generalized coordinates as

$$\frac{\partial \hat{\mathbf{Q}}}{\partial t} + \frac{\partial(\hat{\mathbf{F}} - \hat{\mathbf{F}}_v)}{\partial \xi} + \frac{\partial(\hat{\mathbf{G}} - \hat{\mathbf{G}}_v)}{\partial \eta} + \frac{\partial(\hat{\mathbf{H}} - \hat{\mathbf{H}}_v)}{\partial \zeta} = 0 \tag{A-1}$$

A general, three-dimensional transformation between the Cartesian variables (x, y, z) and the generalized coordinated (ξ, η, ζ) is implied. (See Appendix F for details.) The variable J represents the Jacobian of the transformation:

$$J = \frac{\partial(\xi, \eta, \zeta, t)}{\partial(x, y, z, t)} \tag{A-2}$$

In Equation (A-1), $\mathbf{Q}$ is the vector of conserved variables, density, momentum, and total energy per unit volume, such that

$$\hat{\mathbf{Q}} = \frac{\mathbf{Q}}{J} = \frac{1}{J}\begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ e \end{bmatrix} \tag{A-3}$$

The inviscid flux terms are

$$\hat{\mathbf{F}} = \frac{\mathbf{F}}{J} = \frac{1}{J}\begin{bmatrix} \rho U \\ \rho U u + \xi_x p \\ \rho U v + \xi_y p \\ \rho U w + \xi_z p \\ (e + p)U - \xi_t p \end{bmatrix} \tag{A-4}$$

$$\hat{\mathbf{G}} = \frac{\mathbf{G}}{J} = \frac{1}{J}\begin{bmatrix} \rho V \\ \rho V u + \eta_x p \\ \rho V v + \eta_y p \\ \rho V w + \eta_z p \\ (e + p)V - \eta_t p \end{bmatrix} \tag{A-5}$$

$$\hat{\mathbf{H}} = \frac{\mathbf{H}}{J} = \frac{1}{J}\begin{bmatrix} \rho W \\ \rho W u + \zeta_x p \\ \rho W v + \zeta_y p \\ \rho W w + \zeta_z p \\ (e + p)W - \zeta_t p \end{bmatrix} \tag{A-6}$$

The contravariant velocities are given by

$$\begin{aligned} U &= \xi_x u + \xi_y v + \xi_z w + \xi_t \\ V &= \eta_x u + \eta_y v + \eta_z w + \eta_t \\ W &= \zeta_x u + \zeta_y v + \zeta_z w + \zeta_t \end{aligned} \tag{A-7}$$

The viscous flux terms are

$$\hat{\mathbf{F}}_v = \frac{\mathbf{F}_v}{J} = \frac{1}{J}\begin{bmatrix} 0 \\ \xi_x \tau_{xx} + \xi_y \tau_{xy} + \xi_z \tau_{xz} \\ \xi_x \tau_{xy} + \xi_y \tau_{yy} + \xi_z \tau_{yz} \\ \xi_x \tau_{xz} + \xi_y \tau_{yz} + \xi_z \tau_{zz} \\ \xi_x b_x + \xi_y b_y + \xi_z b_z \end{bmatrix} \tag{A-8}$$

$$\hat{\mathbf{G}}_v = \frac{\mathbf{G}_v}{J} = \frac{1}{J}\begin{bmatrix} 0 \\ \eta_x \tau_{xx} + \eta_y \tau_{xy} + \eta_z \tau_{xz} \\ \eta_x \tau_{xy} + \eta_y \tau_{yy} + \eta_z \tau_{yz} \\ \eta_x \tau_{xz} + \eta_y \tau_{yz} + \eta_z \tau_{zz} \\ \eta_x b_x + \eta_y b_y + \eta_z b_z \end{bmatrix} \tag{A-9}$$

$$\hat{\mathbf{H}}_v = \frac{\mathbf{H}_v}{J} = \frac{1}{J}\begin{bmatrix} 0 \\ \zeta_x\tau_{xx} + \zeta_y\tau_{xy} + \zeta_z\tau_{xz} \\ \zeta_x\tau_{xy} + \zeta_y\tau_{yy} + \zeta_z\tau_{yz} \\ \zeta_x\tau_{xz} + \zeta_y\tau_{yz} + \zeta_z\tau_{zz} \\ \zeta_x b_x + \zeta_y b_y + \zeta_z b_z \end{bmatrix} \quad \text{(A-10)}$$

The shear stress and hear flux terms are defined in tensor notations (summation convention implied) as

$$\tau_{x_i x_j} = \frac{M_\infty}{Re_{\tilde{L}_R}}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda\frac{\partial u_k}{\partial x_k}\delta_{ij}\right] \quad \text{(A-11)}$$

$$b_{x_i} = u_j\tau_{x_i x_j} - \dot{q}_{x_i} \quad \text{(A-12)}$$

$$\dot{q}_{x_i} = -\left[\frac{M_\infty\mu}{Re_{\tilde{L}_R} Pr(\gamma-1)}\right]\frac{\partial a^2}{\partial x_i} \quad \text{(A-13)}$$

The pressure is obtained by the equation of state for a perfect gas

$$p = (\gamma - 1)\left[e - \frac{\rho}{2}(u^2 + v^2 + w^2)\right] \quad \text{(A-14)}$$

The above equations have been nondimensionalized in terms of the free-stream[1] density, $\tilde{\rho}_\infty$, the free-stream speed of sound, $\tilde{a}_\infty$, and the free-stream molecular viscosity, $\tilde{\mu}_\infty$. (See Chapter 4.) The chain rule is used to evaluate derivatives with respect to (x, y, z) in terms of (ξ, η, ζ). Consistent with the thin-layer assumption, only those derivatives in the direction normal to the wall (ζ) are retained in the shear stress and heat flux terms. Equation (A-1) is closed by the Stokes hypothesis for bulk viscosity ($\lambda + 2\mu/3=0$) and Sutherland's law for molecular viscosity.[45]

The CFL3D code also has the capability to solve the Euler equations, which are obtained when the $\hat{\mathbf{F}}_v$, $\hat{\mathbf{G}}_v$, and $\hat{\mathbf{H}}_v$ terms are omitted from Equation (A-1).

The numerical algorithm uses a semi-discrete finite-volume formulation, resulting in a consistent approximation to the conservation laws in integral form

$$\frac{\partial}{\partial t}\iiint_V \mathbf{Q}\,dV + \iint_S \vec{f}\cdot\vec{n}\,dS = 0 \quad \text{(A-15)}$$

[1] See the note on page 3 about the usage of the phrase *free stream*.

where $\vec{f}$ denotes the net flux through a surface S with unit normal $\vec{n}$ containing the (time-invariant) volume V. Integration of Equation (A-15) over a control volume bounded by lines of constant ξ, η, and ζ gives the semi-discrete form

$$\begin{aligned}\left(\frac{\partial \hat{\mathbf{Q}}}{\partial t}\right)_{i,\,j,\,k} &+ (\hat{\mathbf{F}} - \hat{\mathbf{F}}_v)_{i+1/2,\,j,\,k} - (\hat{\mathbf{F}} - \hat{\mathbf{F}}_v)_{i-1/2,\,j,\,k} \\ &+ (\hat{\mathbf{G}} - \hat{\mathbf{G}}_v)_{i,\,j+1/2,\,k} - (\hat{\mathbf{G}} - \hat{\mathbf{G}}_v)_{i,\,j-1/2,\,k} \\ &+ (\hat{\mathbf{H}} - \hat{\mathbf{H}}_v)_{i,\,j,\,k+1/2} - (\hat{\mathbf{H}} - \hat{\mathbf{H}}_v)_{i,\,j,\,k-1/2} = 0\end{aligned} \tag{A-16}$$

where, for convenience,

$$\begin{aligned}\Delta\xi &= \xi_{i+1/2,\,j,\,k} - \xi_{i-1/2,\,j,\,k} = 1 \\ \Delta\eta &= \eta_{i,\,j+1/2,\,k} - \eta_{i,\,j-1/2,\,k} = 1 \\ \Delta\zeta &= \zeta_{i,\,j,\,k+1/2} - \zeta_{i,\,j,\,k-1/2} = 1\end{aligned} \tag{A-17}$$

The discrete values $\hat{\mathbf{Q}}_{i,\,j,\,k}$ are regarded as average values taken over a unit computational cell; similarly, discrete values of $\hat{\mathbf{F}}$, $\hat{\mathbf{G}}$, and $\hat{\mathbf{H}}$ are regarded as face-average values. The convective and pressure terms are differenced using either the upwind flux-difference-splitting technique of Roe[31] or the flux-vector-splitting technique of van Leer.[39] The MUSCL (Monotone Upstream-centered Scheme for Conservation Laws) approach of van Leer[40] is used to determine state-variable interpolations at the cell interfaces. The shear stress and heat transfer terms are centrally differenced.