CHAPTER 4

Nondimensionalization

The goal of many CFD problems is to solve the flow over configurations that have been tested in wind tunnels or flight tests. In experimental tests, the model being studied is of known dimensions. Since there is no standard system of measurement for such testing, some models are measured in feet, some in meters, some in inches, etc. Similarly, wind speeds might be measured in feet/second or meters/second or some other way. Nondimensionalizing the flow-field parameters in a CFD code removes the necessity of converting from one system to another within the code.

Throughout this chapter, the term *free stream* is used. Please see the note on page 3 concerning its usage.

4.1 Dimensional Parameter Definitions

The following parameters are used to nondimensionalize the flow-field variables:

$\tilde{L}$ = characteristic length, feet (e.g. chord)

L_{ref} = corresponding length in the grid (nondimensional)

$\tilde{L}_R = \tilde{L}/L_{ref}$ = reference length used by code, feet (dimensional)

$\tilde{\rho}_\infty$ = free-stream density, slug/feet^3

$\tilde{a}_\infty$ = free-stream speed of sound, feet/second

$|\tilde{\mathbf{V}}|_\infty$ = free-stream velocity, feet/second

$\tilde{\mu}_\infty$ = free-stream molecular viscosity, slug/feet-second

$\tilde{t}$ = time, seconds

The ~ here and elsewhere indicates a dimensional quantity. Note that meters could be substituted for feet, kg for slugs, etc., in the above definitions, as long as it is done consistently.

4.2 General Flow-Field Variables

In CFL3D, the nondimensionalizations are performed as follows:

$$
\begin{aligned}
\rho &= \frac{\tilde{\rho}}{\tilde{\rho}_\infty} & \rho_\infty &= 1 \\
u &= \frac{\tilde{u}}{\tilde{a}_\infty} & u_\infty &= M_\infty \cos\alpha \cos\beta \\
v &= \frac{\tilde{v}}{\tilde{a}_\infty} & v_\infty &= -M_\infty \sin\beta \\
w &= \frac{\tilde{w}}{\tilde{a}_\infty} & w_\infty &= M_\infty \sin\alpha \cos\beta \\
p &= \frac{\tilde{p}}{\tilde{\rho}_\infty (\tilde{a}_\infty)^2} & p_\infty &= \frac{1}{\gamma}
\end{aligned}
\tag{4-1}
$$

Also,

$$
\begin{aligned}
e &= \frac{\tilde{e}}{\tilde{\rho}_\infty (\tilde{a}_\infty)^2} & e_\infty &= \frac{1}{\gamma(\gamma-1)} + \frac{(M_\infty)^2}{2} \\
a &= \frac{\tilde{a}}{\tilde{a}_\infty} & a_\infty &= 1 \\
T &= \frac{\tilde{T}}{\tilde{T}_\infty} = \frac{\gamma p}{\rho} = a^2 & T_\infty &= 1
\end{aligned}
\tag{4-2}
$$

$$
\begin{aligned}
x &= \frac{\tilde{x}}{\tilde{L}_R} & y &= \frac{\tilde{y}}{\tilde{L}_R} \\
z &= \frac{\tilde{z}}{\tilde{L}_R} & t &= \frac{\tilde{t}\tilde{a}_\infty}{\tilde{L}_R}
\end{aligned}
\tag{4-3}
$$

Also, Reynolds number based on the characteristic length is

$$Re_{\tilde{L}} = \frac{\tilde{\rho}_\infty |\tilde{\mathbf{V}}|_\infty \tilde{L}}{\tilde{\mu}_\infty} \tag{4-4}$$

Note that $Re_{\tilde{L}}$ is *not* input into the code, but rather the Reynolds number per unit grid length, in millions:

$$\mathbf{reue} = \frac{Re_{\tilde{L}}}{L_{ref}} \times 10^{-6} = \frac{\tilde{\rho}_\infty |\tilde{\mathbf{V}}|_\infty \tilde{L}_R}{\tilde{\mu}_\infty} \times 10^{-6} = Re_{\tilde{L}_R} \times 10^{-6} \tag{4-5}$$

The free-stream Mach number is

$$M_\infty = \frac{|\tilde{\mathbf{V}}|_\infty}{\tilde{a}_\infty} \tag{4-6}$$

where the free-stream velocity magnitude is

$$|\tilde{\mathbf{V}}|_\infty = \sqrt{(\tilde{u}_\infty)^2 + (\tilde{v}_\infty)^2 + (\tilde{w}_\infty)^2} \tag{4-7}$$

The molecular viscosity is defined from Sutherland's[45] law as follows:

$$\mu = \frac{\tilde{\mu}}{\tilde{\mu}_\infty} = \left(\frac{\tilde{T}}{\tilde{T}_\infty}\right)^{\frac{3}{2}} \left[\frac{\tilde{T}_\infty + \tilde{c}}{\tilde{T} + \tilde{c}}\right] \tag{4-8}$$

$$= (T)^{\frac{3}{2}} \left[\frac{1 + \dfrac{\tilde{c}}{\tilde{T}_\infty}}{T + \dfrac{\tilde{c}}{\tilde{T}_\infty}}\right]$$

where $\tilde{c} = 198.6°R = 110.4°K$ is Sutherland's constant.

4.3 Reynolds Number Examples

Experience has shown that some confusion arises when it comes to setting the parameter **reue** in CFL3D. Some examples may help.

Suppose the configuration to be analyzed has a body length of 48 inches and it is desired to run a simulation in which the Reynolds number based on the body length is 6×10^6. Thus, $\tilde{L} = 48$ inches and $Re_{\tilde{L}} = 6\times10^6$.

To carry out the simulation, a grid is required. Suppose three people each create a grid. In the first person's grid, the length of the body is 48, i.e. the configuration is full size in the grid coordinate system. Thus, L_{ref} in grid 1 is 48. Therefore, by Equation (4-5),

$$\mathbf{reue} = \frac{6\times10^6}{48}\times10^{-6} = 0.125 \tag{4-9}$$

The second person decides to normalize the grid such that the body has a length of 1. Thus, L_{ref} in grid 2 is 1 and by Equation (4-5),

$$\mathbf{reue} = \frac{6\times10^6}{1}\times10^{-6} = 6.0 \tag{4-10}$$

The third person was given the geometry definition in centimeters. Grid 3 is therefore generated with a body length of 121.9 (48 inches ≈ 121.9 centimeters), so $L_{ref} = 121.9$ and

$$\mathbf{reue} = \frac{6\times10^6}{121.9}\times10^{-6} = 0.04922 \tag{4-11}$$

4.4 Time and Time Step

Time is nondimensionalized as follows:

$$t = \frac{\tilde{t}\tilde{a}_\infty}{\tilde{L}_R} \tag{4-12}$$

If a particular dimensional time step (**dt** in "LT5 - Time Step Parameters" on page 21) is required, then this relation may be used to determine the corresponding input value of Δt.

For the time step,

$\tilde{\Delta t}$ = desired time step, seconds

Therefore,

$$\Delta t = \frac{\tilde{\Delta t}\tilde{a}_\infty}{\tilde{L}_R} \tag{4-13}$$

4.5 Moving Mesh

These additional values may be needed for the nondimensionalization of the moving-mesh parameters:

$\tilde{f}$ = frequency, cycles/second

k_r = reduced frequency, $k_r = \tilde{f}\tilde{L}/\tilde{a}_\infty$ Note: This definition of reduced frequency differs from the “standard” definition $k_r = \tilde{f}\tilde{L}/|\tilde{\mathbf{V}}|_\infty$. Also, note that the quantities $\tilde{L}$ and L_{ref} used to nondimensionalize the moving mesh data need not be the *same* as those used to nondimensionalize x, y, z, and t. That is why L_{ref} is input explicitly in the moving mesh section of the input file.

$k_g = \tilde{L}/(2\pi\tilde{a}_\infty\tilde{t}_0)$ = rate of growth of displacement, where $\tilde{t}_0$ = the time to reach 63.2% of the maximum displacement, seconds

4.5.1 Translational Motion

These dimensional parameters are also needed for the translational nondimensionalizations:

$\tilde{u}_{trans}$ = x component of translational velocity, feet/second (needed for **itrans** = 1)

$\tilde{v}_{trans}$ = y component of translational velocity, feet/second (needed for **itrans** = 1)

$\tilde{w}_{trans}$ = z component of translational velocity, feet/second (needed for **itrans** = 1)

$\tilde{x}_{max}$ = maximum x displacement, feet (needed for **itrans** > 1)

$\tilde{y}_{max}$ = maximum y displacement, feet (needed for **itrans** > 1)

$\tilde{z}_{max}$ = maximum z displacement, feet (needed for **itrans** > 1)

Again note that meters or inches, etc. could be substituted for feet in the above definitions, as long as it is done consistently. Also, not all of the above dimensional parameters are relevant to a given problem. For example, in sinusoidal plunging (**itrans** = 2), k_g, $\tilde{u}_{trans}$, $\tilde{v}_{trans}$, and $\tilde{w}_{trans}$ are irrelevant.

4.5.1.1 Translation With Constant Velocity

For **itrans** = 1, input the following nondimensional parameters:

rfreq = 0.0

utrans = $\tilde{u}_{trans}/\tilde{a}_{\infty}$ (Mach number of x translation)

vtrans = $\tilde{v}_{trans}/\tilde{a}_{\infty}$

wtrans = $\tilde{w}_{trans}/\tilde{a}_{\infty}$

The nondimensional displacement in the code is then governed by

$$\begin{aligned} x &= u_{trans}t \\ y &= v_{trans}t \\ z &= w_{trans}t \end{aligned} \tag{4-14}$$

4.5.1.2 Sinusoidal Plunging

For **itrans** = 2, input the following nondimensional parameters:

rfreq = k_r

utrans = $\tilde{x}_{max}/\tilde{L}_R$

vtrans = $\tilde{y}_{max}/\tilde{L}_R$

wtrans = $\tilde{z}_{max}/\tilde{L}_R$

Note that, $\tilde{x}_{max}/\tilde{L}_R$, for example, is simply the maximum x displacement in “grid units”. The nondimensional displacement in the code is then governed by

$$x = u_{trans}\sin\left(2\pi\frac{k_r t}{L_{ref}}\right)$$
$$y = v_{trans}\sin\left(2\pi\frac{k_r t}{L_{ref}}\right) \quad (4\text{-}15)$$
$$z = w_{trans}\sin\left(2\pi\frac{k_r t}{L_{ref}}\right)$$

If the time step is to be chosen so that there are N steps per cycle, the required input value of **dt** (see “LT5 - Time Step Parameters” on page 21) may be determined from

$$\mathbf{dt} = \frac{L_{ref}}{k_r \cdot N} \quad (4\text{-}16)$$

The corresponding dimensional time step is

$$\tilde{\Delta t} = \frac{1}{N \cdot \tilde{f}} \quad (4\text{-}17)$$

4.5.1.3 Acceleration to Constant Displacement

For **itrans** = 3, input the following nondimensional parameters:

rfreq = k_g

utrans = $\tilde{x}_{max}/\tilde{L}_R$

vtrans = $\tilde{y}_{max}/\tilde{L}_R$

wtrans = $\tilde{z}_{max}/\tilde{L}_R$

Note that, $\tilde{x}_{max}/\tilde{L}_R$, for example, is simply the maximum x displacement in “grid units”. The nondimensional displacement in the code is then governed by

$$
\begin{aligned}
x &= u_{trans}\left[1.0 - \exp\left(-2\pi\frac{k_r t}{L_{ref}}\right)\right] \\
y &= v_{trans}\left[1.0 - \exp\left(-2\pi\frac{k_r t}{L_{ref}}\right)\right] \\
z &= w_{trans}\left[1.0 - \exp\left(-2\pi\frac{k_r t}{L_{ref}}\right)\right]
\end{aligned}
\tag{4-18}
$$

Thus, with **utrans** = x_{max}, x will attain 63.2% of it's maximum value, i.e.

$$\frac{x}{x_{max}} = 1 - \frac{1}{\mathrm{e}} \tag{4-19}$$

at a nondimensional time of

$$t_0 = \frac{L_{ref}}{2\pi k_r} \tag{4-20}$$

rfreq for **itrans** = 3 may therefore be defined as

$$\mathbf{rfreq} = \frac{L_{ref}}{2\pi t_0} = \frac{\tilde{L}}{2\pi\tilde{a}_\infty\tilde{t}_0} \tag{4-21}$$

where $\tilde{t}_0$ is the time (seconds) to reach 63.2% of the maximum displacement. Increasing **rfreq** has the effect of decreasing the time to attain a specified percentage of the asymptotic maximum displacement.

4.5.2 Rotational Motion

These additional dimensional parameters are needed for the rotational nondimensionalizations:

$\tilde{\omega}_x$ = x component of rotational velocity, revolutions/second (needed for **irotat** = 1)

$\tilde{\omega}_y$ = y component of rotational velocity, revolutions/second (needed for **irotat** = 1)

$\tilde{\omega}_z$ = z component of rotational velocity, revolutions/second (needed for **irotat** = 1)

$\tilde{\theta}_{x,\,max}$ = maximum angular displacement about x axis, degrees (needed for **irotat** >1)

$\tilde{\theta}_{y,\,max}$ = maximum angular displacement about y axis, degrees (needed for **irotat** >1)

$\tilde{\theta}_{z,\,max}$ = maximum angular displacement about z axis, degrees (needed for **irotat** >1)

Again note that meters or inches, etc. could be substituted for feet in the above definitions, as long as it is done consistently. Not all of the above dimensional parameters are relevant to a given problem. For example, in sinusoidal pitching (**irotat** = 2) k_g, $\tilde{\omega}_x$, $\tilde{\omega}_y$, and $\tilde{\omega}_z$ are irrelevant.

Using these dimensional values, the nondimensionalizations are performed as follows:

4.5.2.1 Rotation With Constant Angular Velocity

For **irotat** = 1, input the following nondimensional parameters:

rfreq = 0.

omegax = $\tilde{\omega}_x \tilde{L} / \tilde{a}_\infty$

omegay = $\tilde{\omega}_y \tilde{L} / \tilde{a}_\infty$

omegaz = $\tilde{\omega}_z \tilde{L} / \tilde{a}_\infty$

For turbomachinery type applications, $\tilde{L}$ is typically chosen as the prop/blade diameter, so that $\tilde{\omega}_x$, $\tilde{\omega}_y$, $\tilde{\omega}_z$ = M_∞ / (advance ratio). The angular displacement (radians) in the code is then governed by

$$\begin{aligned} \theta_x &= 2\pi\frac{\omega_x t}{L_{ref}} \\ \theta_y &= 2\pi\frac{\omega_y t}{L_{ref}} \\ \theta_z &= 2\pi\frac{\omega_z t}{L_{ref}} \end{aligned} \qquad \text{(4-22)}$$

4.5.2.2 Sinusoidal Variation of Angular Displacement

For **irotat** = 2, input the following parameters:

rfreq $= k_r$

omegax = $\tilde{\theta}_{x,\,max}$, degrees

omegay = $\tilde{\theta}_{y,\,max}$, degrees

omegaz = $\tilde{\theta}_{z,\,max}$, degrees

The rotational displacement (radians) in the code is then governed by

$$\begin{aligned}\theta_x &= \omega_x \frac{\pi}{180} \sin\left(2\pi k_r \frac{t}{L_{ref}}\right) \\ \theta_y &= \omega_y \frac{\pi}{180} \sin\left(2\pi k_r \frac{t}{L_{ref}}\right) \\ \theta_z &= \omega_z \frac{\pi}{180} \sin\left(2\pi k_r \frac{t}{L_{ref}}\right)\end{aligned} \tag{4-23}$$

If the time step is to be chosen so that there are N steps per cycle, the required input value of **dt** (see “LT5 - Time Step Parameters” on page 21) may be determined from

$$\Delta t = \frac{L_{ref}}{k_r N} \tag{4-24}$$

The corresponding dimensional time step is:

$$\tilde{\Delta} t = \frac{1}{N\tilde{f}} \tag{4-25}$$

4.5.2.3 Acceleration to a Constant Rotational Displacement

For **irotat** = 3, input the following nondimensional parameters:

rfreq $= k_g$

omegax = $\tilde{\theta}_{x,\,max}$, degrees

omegay = $\tilde{\theta}_{y,\,max}$, degrees

omegaz = $\tilde{\theta}_{z,\,max}$, degrees

The rotational displacement (radians) in the code is then governed by

$$\theta_x = \frac{\pi\omega_x}{180}\left[1 - \exp\left(-2\pi k_r \frac{t}{L_{ref}}\right)\right]$$
$$\theta_y = \frac{\pi\omega_y}{180}\left[1 - \exp\left(-2\pi k_r \frac{t}{L_{ref}}\right)\right] \quad \text{(4-26)}$$
$$\theta_z = \frac{\pi\omega_z}{180}\left[1 - \exp\left(-2\pi k_r \frac{t}{L_{ref}}\right)\right]$$

Thus, with $\omega_x = \tilde{\theta}_{x,\,max}$, θ_x will attain 63.2% of it's maximum value, i.e.

$$\theta_x / \theta_{x,\,max} = 1 - \frac{1}{e} \quad \text{(4-27)}$$

at a nondimensional time of

$$t_0 = \frac{L_{ref}}{2\pi k_r} \quad \text{(4-28)}$$

Increasing k_r has the effect of decreasing the time to attain a specified percentage of the asymptotic maximum displacement. **rfreq** for **irotat** = 3 may therefore be defined as

$$\mathbf{rfreq} = \frac{L_{ref}}{2\pi t_0} = \frac{\tilde{L}}{2\pi \tilde{a}_\infty \tilde{t}_0} \quad \text{(4-29)}$$

where $\tilde{t}_0$ is the time (seconds) to reach 63.2% of maximum displacement.